NUMBERS

L.F.Taylor

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FABER

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L. F. TAYLOR

'Curiosity is one of the few virtues common to all scientists but unfortunately today its satisfaction is mostly dependent upon expensive equipment and closely knit team work, says Mr. Taylor in his preface. 'The research laboratory of a Theory of Numbers enthusiast can, at a pinch, be equipped with nothing more than pencil and paper. And to anyone who hesitates to set up such a laboratory and embark upon his own research because of the view that it has all been done before and there is nothing left to discover I would say with emphasis that this is not true. In mathematics there is always another field beyond the hedge if one can only find the gap to crawl through.'

In *Numbers* Mr. Taylor suggests some ways in which the gaps may be found and encourages his readers to discover what lies beyond them, and the result is a book which everyone who is drawn to numbers and the problems that arise from them will find fascinating.

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L. F. TAYLOR

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PREFACE

I don't know whether many people ever bother to read prefaces but I feel that anyone who wishes to find what a book is about should at least be offered the opportunity at as early a stage as possible.

First I must make it clear that, although I hope there is something to be learned from this one, it is in no sense a 'teaching' book: its intended purpose is to interest potential mathematicians and to introduce them to a branch of arithmetic which may transform their schoolday impressions of the subject.

Curiosity is one of the few virtues common to all scientists but unfortunately today its satisfaction is mostly dependent upon expensive equipment and closely knit team work. The research laboratory of a Theory of Numbers enthusiast can, at a pinch, be equipped with nothing more than pencil and paper. And to anyone who hesitates to set up such a laboratory and embark upon his own research because of the view that it has all been done before and there is nothing left to discover I would say with emphasis that this is not true. In mathematics there is always another field beyond the hedge if one can only find the gap to crawl through.

The main themes of this book are 'Factorisation' and 'Prime Numbers' but there are some diversions into such topics as recurring decimals, additive series, a brief note on Diophantine equations and a few others. They have all been chosen because they are among those branches of Number Theory which I have found intensely interesting—and I hope some of this enthusiasm will rub off during the reading—and also because they show how unexpectedly some divergent lines of enquiry latch in together.

I have included a chapter (Finite Arithmetic) on elementary congruence technique since some knowledge of this brilliant invention is essential to any serious study of large numbers. For the same reason there is a brief note on the 'Converse of Fermat's Theorem' and its relation to composite numbers.

A number of tables of values and functions have been included in the text and the Appendix; these have the double service of providing examples for the former and at the same time saving the potential research worker a certain amount of muscular mathematics.

Most readers will, I am sure, find something about the properties of numbers which is new to them in the following pages. (How many professionals, for instance, know of a direct linkage between 1/79—having a recurring period of thirteen digits—and $\sqrt{2}$ which is irrational?)

However it is the amateur mathematician I have had most in mind in compiling the following notes, and in defining my conception of the term 'amateur' I cannot do better than quote J. E. Littlewood (*A Mathematician's Miscellany*, Methuen & Co. Ltd.).

'I constantly meet people who are doubtful, generally without due reason, about their potential capacity . . . If your education just included, or just stopped short of including, 'a little calculus', you are fairly high in the amateur class.'

Coming from such a master this is pretty heartening and I hope his words will encourage those who find enjoyment and relaxation in solving puzzles to apply their ingenuity to experimenting with numbers.

Finally I would like to stress that in studies of this nature it is particularly important that the reader takes more than a passive interest in the text and I have therefore ended each chapter with a few relevant exercises. (The answers will be found at the end of the book.)

I am greatly indebted to Mr. Nicholas E. Scripture for some valuable advice and a critical scrutiny of the manuscript.

I COUNTING

(1) In this book we shall be concerned almost entirely with the integers. Those all too familiar numbers 1, 2, 3, . . . which go on interminably, each one formed by adding 'one' to the last.

When man first began to feel the need for more specific terms than his current equivalent sounds for 'a', 'few' and 'many' he began to count. And counting is the basis of all arithmetic for it involves the process of addition.

One can only speculate on the evolutionary processes that led up to the system of counting by the addition of units as we know it, or on the many experiments which must have been tried before we settled down to counting generally in bundles of ten units. It has been plausibly put forward that the decimal scale originated in our having ten fingers on which to count but the weights and measures and the monetary systems of the world give this view little support.

Be that as it may we now have a stable and familiar arithmetic which uses the ten symbols, 0, 1, 2, . . . 9, and expresses numbers greater than nine in a form of shorthand based on the powers of ten. Thus the number 'two thousand five hundred and seventy nine' contains $9 \times 10^0 + 7 \times 10^1 + 5 \times 10^2 + 2 \times 10^3$ units and is conventionally written 2579.

(In the power index notation $n^0 = 1$, and $n^1 = n$.) If 2579 is successively divided by ten the remainders 9, then 7, 5, and 2 are left, these being its own digits reading from right to left.

All very elementary of course, but only because we are so accustomed to this system that there is no longer any need to stop and think how the numbers we use are constructed. Their expression as multiples of powers of ten is, as we shall see, well chosen but it is none the less a matter of pure convention and it should not be forgotten that the same numbers could just as well be expressed as multiples of powers of any base we care to choose.

In order to convert a number in the decimal scale to that of any

other base it is only necessary to divide successively by the number of the new base writing down the remainders in the reverse order to that in which they appear. Converting the number 2579 into the scale of seven for instance we have:

	Remainder
7)2579	3
7) 368	4
7) 52	3
7) 7	0
1	1

the new expression now being written 10343, indicating that it is equal to:

$$
\begin{aligned}
3 \times 7^0 &= 3 \times 1 = 3 \\
4 \times 7^1 &= 4 \times 7 = 28 \\
3 \times 7^2 &= 3 \times 49 = 147 \\
0 \times 7^3 &= 0 \\
1 \times 7^4 &= 1 \times 2401 = 2401 \\
& \qquad \qquad \quad 2579
\end{aligned}
$$

It will be seen that in dividing by seven there can be no remainder greater than six and so the only digits that appear in this scale are 0, 1, 2, . . . 6. Similarly in the scale of two we require only the two digits 0 and 1.

For example:

2)2579	1
2)1289	1
2) 644	0
2) 322	0
2) 161	1
2) 80	0
2) 40	0
2) 20	0
2) 10	0
2) 5	1
2) 2	0
1	1

the decimal number 2579 now appearing as 101000010011 which states explicitly that it is equal to

$$2^0 + 2^1 + 2^4 + 2^9 + 2^{11}$$

In just the same way the number could be given in powers of any number greater than ten, say for example twenty-three. (The divisions by 23 are given here in condensed form.)

23)2579	3
23) 112	20
4	4

In writing this down we have to remember that the '20' must be regarded here as a single digit; this can be made clear by enclosing it in brackets and the new number now appears as 4(20)3 indicating that 2579 is also equal to $3 + (20 \times 23) + (4 \times 23^2)$.

It will be convenient to denote any arrangement of digits as N_s, in which s represents the scale of notation employed: the observations that have been made can then be written concisely as follows,

$$
\begin{aligned}
(N_{10})\, 2579 &= (N_7)\, 10343 \\
&= (N_2)\, 101000010011 \\
&= (N_{23})\, 4(20)3
\end{aligned}
$$

It is now seen that whilst the choice is quite arbitrary the 'powers of ten' notation presents a good all-round middle course for expressing numbers requiring more than one digit; the use of fewer symbols calls for longer strings of digits to represent a given number while more symbols would have meant more and longer multiplication tables to memorise.

(2) There are occasions, however, when scales other than the decimal can be used with advantage. One of these will have become familiar to readers through the widespread development of electronic digital computers. Here the binary system which employs the two digits 0 and 1, is readily correlated with the two (on-off) positions of an electrical switch whilst an increase in the number of digits is of small consequence.

Again, because the binary system's largest digit is 1, it follows that any number whatever can be expressed as the sum of single powers of 2. Thus it is a simple matter to multiply together any two numbers without going outside the $\times 2$ multiplication table.

This is done quite easily by successively multiplying one of the numbers and dividing the other by two and then deleting those numbers in the increasing column which correspond with the even numbers in the other.

In effect we are thus finding by successive division those powers of two which are needed to represent one number, multiplying the other by $2, 2^2, 2^3,$ etc., and then rejecting the products which are not required.

For example: To find $37 \times 127 (= 4699)$.

Divide by 2		Multiply by 2
37	1	127
18	0	~~254~~
9	1	508
4	0	~~1016~~
2	0	~~2032~~
1	1	4064
		4699 = Answer.

The left hand column will be familiar; in it we have converted (N_{10}) 37 to (N_2) 100101 showing that

$$37 = 2^0 + 2^2 + 2^5 = 1 + 4 + 32.$$

The right hand column is made up of 127 multiplied in turn by $2^0, 2^1, 2^2, \ldots 2^5$, and from it we have deleted the products of $2, 2^3,$ and 2^4, the remaining figures then being added. In practice there is no need to write down the first column remainders since the zeros occur alongside even numbers and these can consequently be used to locate the deletion lines.

The process is much simpler to carry out than to describe and it is rather surprising that it is not taught as a rule of thumb method in those schools whose children have difficulty with their multiplication tables.

(3) Apart from these instances one might readily conclude that there is little to be gained by any further study of scale-changing and indeed this is about as far as most textbooks take the subject. Nevertheless it is never safe to assume that any branch of mathematics

has arrived at a dead-end and it will be seen that further developments are not only possible but of practical interest. But first it will be necessary to make a digression over well trodden ground.

There are two well known and simple tests for determining whether a given number is divisible by nine or eleven. For the former the digits of the number are added together to form another number and if this contains more than one digit the process is continued until only one remains. If this digit is 9 then the original number is exactly divisible by 9; any other resultant digit will be the remainder found on dividing the original number by 9. Thus the number 23456723 when divided by nine will leave the remainder 5, since $2 + 3 + 4 + 5 + 6 + 7 + 2 + 3 = 32$, and $3 + 2 = 5$. The rule, commonly known as 'casting out the nines', can also be applied to division by three, the remainder in the above case then being $5 - 3 = 2$.

Somewhat similarly a number is exactly divisible by eleven if the difference between the sums of its alternate digits is 0 or a multiple of eleven. For example 73645 is divisible by eleven since $(5 + 6 + 7) - (4 + 3) = 11$.

Now the numbers nine and eleven are respectively $10 - 1$, and $10 + 1$ and it would be reasonable to expect that the same rules apply to the divisors $s - 1$ and $s + 1$ when dealing with numbers expressed in the scale of s. Suppose for example we take the number $4571 (= 7 \times 653)$ and transpose it into the scale of 8, thus:

8	4571	3
	571	3
	71	7
	8	0
	1	1

Thus (N_{10}) 4571 = (N_8) 10733.
And since in the scale of 8 the 'nines' rule applies to division by 7 we have $3 + 3 + 7 + 1 = 14 = 2 \times 7$, and therefore 7 is a divisor of 4571.

(We could have continued, $14 = (N_8)$ 16. And $6 + 1 = 7$)

Or again, converting to the scale of six and using the 'eleven' rule we have:

$$(N_{10})\ 4571 = (N_6)\ 33055$$

and as $5 + 0 + 3 = 5 + 3$ it is shown that 4571 is divisible by $(6 + 1) = 7$.

The principle is in fact quite general and can be proved without difficulty. To take, for simplicity's sake, a specific example, in the decimal scale every number can be expressed as follows:

$a + 10b + 100c + 1000d + \ldots = a + b + 9b + c + 99c + d + 999d + \ldots = a + b + c + d + \ldots + 9m$ (a multiple of 9)

That is, the sum of the digits $a, b, c,$ etc. will be the remainder after dividing by nine.

The 'eleven' rule and a generalisation into any scale of notation can be deduced in an exactly similar manner.

(4) We can now begin to glimpse another practical use for the process of scale changing. In many mathematical operations it is often desirable to know whether a given number is prime, or if not what are its factors. Provided the number is not too large the straightforward approach is to make tentative divisions by the successive primes 7, 11, 13, 17, etc., until a factor is found or the square root of the number is reached. (Obviously if there is no factor less than the square root there cannot be one which is greater and the number must be prime.) From what we have seen above it will be clear that when working through the primes in this way there will occur many cases when two tests can be carried out by making a single scale change. Thus whenever two primes differ by two, that is when they are of the form $n - 1$ and $n + 1$ they can be tested simultaneously by changing the given number into the scale of n and applying the 'nine' and 'eleven' rules to the new digits.

Let us suppose for example that in attempting to factorise the number 18383 all the primes up to and including 23 have been tested without a factor appearing. Now instead of continuing with trial divisions first by 29 and then by 31 we convert the number into the scale of 30, thus:

$$\begin{array}{c|r|r} 30 & 18383 & 23 \\ & 612 & 12 \\ & 20 & 20 \end{array}$$

Applying the two tests to the 'digits' of the new scale we have,

(*a*) $23 + 12 + 20 = 55$

(*b*) $23 + 20 - 12 = 31$

Since 55 is not a multiple of 29 then this is not a factor but the 'eleven' rule shows that 31 is. Dividing out we find that $18383 = 31 \times 593$ and as 31^2 is greater than 593 the latter is also prime.

(5) Now whilst it is possible to test both the prime divisors 41 and 43 in one operation, the conversion into the scale of 42 is 'awkward' and it is doubtful whether much time would be saved. The method is clearly most effective when, as in the last example the divisors to be tested lie on either side of a multiple of ten (i.e. $10n \pm 1$).

But, just as 'casting out the nines' can be used to detect multiples of three—a factor of nine—the above method is equally applicable to primes which are factors of numbers of the forms $10n - 1$ and $10n + 1$. For instance the scale of fifty provides an immediate test for the divisors 7 and 17 because $49 = 7^2$, and $51 = 17 \times 3$.

Taking for example the number 12733 (which equals $7 \times 17 \times 107$) and converting to the scale of fifty, we have:

$$\begin{array}{crr|r} 50 & & 12733 & 33 \\ & & 254 & 4 \\ & & 5 & 5 \end{array}$$

and we see that

$$33 + 4 + 5 = 42 = \text{a multiple of } 7$$
$$33 + 5 - 4 = 34 = \text{a multiple of } 17$$

This property leads at once to an extremely simple test for prime, or prime factors of, numbers of the form $10^n \pm 1$. In these cases no actual division is required to make the scale change as, for example $1317459 = (N_{1000})$, 1(317)(459). Now 999 is a multiple of 37, and $1001 = 7 \times 11 \times 13$ and

since $459 + 317 + 1 = 777 = \underline{37} \times 21$

and $459 + 1 - 317 = 143 = \underline{11} \times \underline{13}$

it is seen at once that 1317459 is exactly divisible by 11, 13 and 37, but not by 7.

In general a given number can be tested for divisibility by any of the factors of $10^n \pm 1$ by separating it into sets of n digits from the right and then applying the 'nine' and 'eleven' rules to these sets.

Incidentally it will now be seen that the basic test for, say, divisibility by eleven is by no means the only one. In fact, since the successive numbers $10 + 1$, $100 - 1$, $1000 + 1$, $10000 - 1$, and so on are all multiples of eleven it is a simple matter to devise a test tailored

to suit the magnitude of any given number. Briefly, the number is divided into large 'sets' and then, according to the number of digits per set the 'nine' or 'eleven' rules are applied until only a two digit number remains. If this is divisible by 11 then so is the original number.

Thus if we wish to find the remainder when the number 7970684847 is divided by 11 we proceed first with subtraction because the number can be separated into sets of five digits:

$$\begin{array}{r} 84847 \\ 79706 \\ \hline 5141 \end{array}$$

and then by addition of pairs of digits:

$$\begin{array}{r} 51 \\ 41 \\ \hline 92 \end{array} = (11 \times 8) + 4.$$

And therefore the remainder is 4.

(6) It will be convenient to draw up a table of 'easy' scale-changes showing the prime numbers to which the $(s-1)$ and $(s+1)$ rules can be applied as tests of divisibility. By 'easy' I mean those scales which do not in effect require successive division by any number greater than eleven.

After what has been said the following Table 1 should be self explanatory; the numbers in brackets are all primes whose divisibility can be tested by changing the given number into the scale 's', and it will be seen that all the primes less than 100 are covered with the exception of 73, 83, and 97. And with some of them a single scale-change may eliminate as many as four possible prime factors.

Provided we remember which column we are working in it is quite practicable to change horses in mid-stream. Suppose, for instance, it is required to find the remainder, if any, on dividing 123456 by 17, then we have,

$$\begin{array}{r|r|r} 800 & 123456 & 256 \\ & 154 & 154 \\ \hline & & 410 \end{array} \qquad \begin{array}{r|r|r} 50 & 410 & 10 \\ & 8 & 8 \\ \hline & & 2 = \text{remainder} \end{array}$$

The exceptions mentioned, namely 73, 83 and 97, could be

Table 1

Scale(s)	$s-1$	$s+1$
10	9 (3)	(11)
20	(19)	21 (7)
30	(29)	(31)
40	39 (13)	(41)
50	49 (7)	51 (17)
60	(59)	(61)
70	69 (23)	(71)
80	(79)	81
90	(89)	91 (7, 13)
100	99 (11)	(101)
110	(109)	111 (37)
200	(199)	201 (67)
300	299 (13, 23)	301 (7, 43)
800	799 (17, 47)	801 (89)
900	899 (29, 31)	901 (17, 53)
1000	999 (37)	1001 (7, 11, 13)

dealt with in the scales of 220, 250 and 290 respectively but these do not present such simple divisors as the above.

EXERCISES

1. Express the integer 11111 in the scale of 2.
2. Express the integer 11111 in the scale of 16.
3. Express the integer 11111 in the scale of 32.
4. What observations can you find to make on the answers to exercises 1, 2, 3, ?
5. Multiply 123 by 456 by the process shown in the text and check by reversing the procedure.
6. Without using ordinary long division find what remainder, if any, is left after dividing $N = 455331$ by 707.

2 CASTING OUT THE PRIMES

(1) In the last chapter we saw how it was possible to tell whether a given number was a multiple of certain primes by dividing it, in effect, by a smaller number. The method was developed by generalising two well-known arithmetical tricks. Let us look at these two again from another angle.

Numbers in the decimal scale which are exactly divisible by nine are recognisable at once, as shown in *1*, 3), from the fact that the sum of their digits is always nine or some multiple of nine, thus:

$$2 \times 9 = 18 \quad \text{and} \quad 8 + 1 = 9$$
$$3 \times 9 = 27 \quad \text{and} \quad 7 + 2 = 9$$
$$4 \times 9 = 36 \quad \text{and} \quad 6 + 3 = 9$$
$$29773 \times 9 = 267957 \quad \text{and} \quad 7 + 5 + 9 + 7 + 6 + 2 = 36$$

and so on.

The above test deals with the number $10t - 1$, where $t = 1$. It would be natural to expect that similar rules could be devised for those numbers where t takes the values 2, 3, 4, . . ., and it is readily seen that this is indeed the case.

Taking, for instance, some multiples of 19, where $t = 2$, and operating upon them by first multiplying the 'units' digit by two and then adding the result to the 'tens' digit it is found that the new number is then either 19 or a smaller multiple of 19. In the latter case the operation is repeated as in the method of 'casting out the nines'.

38	57	76	247	84493 (= 19 × 4447)
16	14	12	14	6
19	19	19	38	8455
			16	10
			19	855
				10
				95
				10
				19

Similarly when $t = 3$, and it is desired to test for multiples of 29, the multiplier now becomes 3 and we have:

58	87	3393	12673 (= 29 × 437)
24	21	9	9
29	29	348	1276
		24	18
		58	145
		24	15
		29	29

The principle is clearly general and can perhaps best be explained by re-examining the procedure when $t = 2$. What we did, in effect, was to multiply the units digit by 19, or $(20 - 1)$, add, and then divide by 10.

For example

38	
152	$= 8 \times 19$
19	

Now let us suppose that $N = 19m = a + 10b$. Then, adding $19a$, we have $19a + a + 10b = 10(2a + b)$. Since 10 is not a divisor of 19 then $2a + b$ must be another multiple of 19 less than $19m$ and if the process is continued we must eventually arrive at the smallest possible multiple, namely 19 itself.

Looking again at the examples given above it will be seen that when N is exactly divisible by 19 the process discloses the quotient m. For instance, in the case of $N = 247$ we have:

247		$247 = 19m$
14		$133 = 19 \times 7$
38	is 'shorthand' for	380
16		$1520 = 19 \times 80$
19		$1900 = 19 \times 100$

Therefore $\quad 19m + (19 \times 7) + (19 \times 80) = 19 \times 100$

or $\quad m = 100 - 87 = 13$

In the algorithm the underlined digits, 87, provide us with m after subtraction from 100.

Similarly when $N = 84493$ the terminal digits show that since

$$10000 - 5553 = 4447, \text{ then } 84493 = 19 \times 4447.$$

Or again, where we were dealing with multiples of 29, in the case of $N = 12673$, the terminal digits are 563.

Then $1000 - 563 = 437$ and hence $12673 = 29 \times 437$.

(2) It is important at this stage to note that nothing has been said to imply that this method of division only has its uses when $10t - 1$ is a prime. In fact it can be applied with the same advantage to any of the prime factors of $10t - 1$, as the following examples will show.

Since $13 \times 3 = 39$, we can use 4 as the multiplier to show that 4771 is a multiple of 13 thus:

$$\begin{array}{r} 4771 \\ 4 \\ \hline 481 \\ 4 \\ \hline 52 \\ 8 \\ \hline 13 \end{array}$$

And, multiplying the terminal digits 211 by 3 and subtracting from 1000 we have the quotient 367.

Similarly, noting that $13 \times 23 = 299$, it will be clear that, using the multiplier 30 as above, it is possible to test both these factors simultaneously.

Given $N = 12029$ $(= 23 \times 523)$ then,

$$\begin{array}{r} 12029 \\ 270 \\ \hline 1472 \\ 60 \\ \hline 207 \\ 210 \\ \hline 23 \end{array}$$

Thus 23 is a factor of 12029 but 13 is not. The quotient can be found as in the previous example.

(3) It is more than likely that the reader is already anticipating the next step in developing these principles. If this is so it might be a good thing for him to close the book at this point—temporarily I hope—and conduct some experiments of his own.

Whether this suggestion is acted upon or not our experience of the effects of scale-changing make it imperative to examine the treatment of those divisors which have, or can be put into, the form of $10t + 1$. As one might expect the method of test depends again upon the value of t, and it is quickly seen that by multiplying by t, etc., as in *2*.1, but in this case *subtracting*, any multiple of $10t + 1$ finally yields the digit 0.

Thus when $t = 2$ we have:

$$\begin{array}{rrl} 42 & 105 & 30639\ (= 21 \times 1459) \\ 4 & 10 & 18 \\ \cline{1-3} 0 & 0 & 3045 \\ & & 10 \\ \cline{3-3} & & 294 \\ & & 8 \\ \cline{3-3} & & 21 \\ & & 2 \\ \cline{3-3} & & 0 \end{array}$$

What is more, as the last example shows, the terminal digits 1 4 5 9 indicate the quotient directly.

As before the test is equally applicable to the factors of $10t + 1$ when this number is composite. In such cases, however, the method may produce a small multiple of the prime under test instead of zero, and the quotient is then rather more elusive. In order to keep this chapter within reasonable bounds I shall give only one example of this particular condition and the reader will be left to discover a general process for finding the quotient.

Let $N = 321433$ $(= 47 \times 6839)$. Now $47 \times 3 = 141$, and we shall use the 'multiplier' 14, thus:

$$\begin{array}{r} 321433 \\ 42 \\ \hline 32101 \\ 14 \\ \hline 3196 \\ 84 \\ \hline 235 \\ 70 \\ \hline -47 \end{array}$$

This example has been given not only to illustrate the above statement but also to show that in many cases this method of testing possible divisors of N has a distinct advantage over the scale-changing technique. As we saw in Chap. 1 the latter is at its best when it can be applied to primes related to what I have called 'easy' divisors. On the other hand, in this system however large we have to make t it has only to be multiplied by the terminal digits, none of which of course can be greater than 9.

(4) As we shall see later, when it comes to the factorisation of large numbers it is important to eliminate as many as possible of the small prime factors before starting on more sophisticated techniques. In the ordinary way this is a laborious process and any method of shortening the work is to be welcomed. The algorithm we have been experimenting with provides such a method and it will now be convenient to gather together our findings into a form available for easy reference.

To test whether a given number is exactly divisible by a prime p, choose a suitable multiple of p such that

$$mp = 10t - 1 \quad \text{or} \quad 10t + 1$$

CASE A *Where* $mp = 10t - 1$.

The number to be tested (N) will, in the decimal scale be of the form $N = 10b + a$. From this derive the number $N_1 = b + ta$. N_1 then equals $10d + c$, and again from this derive $N_2 = d + tc$. Continue in this manner and if eventually a number N_n is reached which equals either p or a small multiple of p then p is an exact divisor of N.

CASE B *Where* $mp = 10t + 1$.

Again given that $N = 10b + a$, the numbers $N_1 = b - ta$, $N_2 = d - tc$, and so on are formed successively as above. If eventually it is found that $N_n = 0$, or p, or a small multiple of p, then p is an exact divisor of N.

This sounds, and probably is, an unnecessarily formal way of describing a process which is extremely easy to operate in practice. This is a difficulty which is bound to arise when one employs methods which do not appear in ordinary arithmetic and is the principle reason why so many examples have been presented.

Now whilst m can be chosen so that pm is equal to either of the

numbers $10t \pm 1$ it will be obvious that for the purpose we have in mind one of these will be preferable to the other. For instance, if we wish to test whether 23 is a divisor then we have $23 \times 3 = 69$, and $t = 7$. To find a multiple of 23 of the form $10t + 1$ we would have to take $23 \times 7 = 161$, t then becoming 16. Neither of these values presents any difficulty but I think most people would choose 7 as the multiplier and the 'addition' process for the test.

In the following table the primes below one hundred are listed together with their appropriate testing multipliers $t(+)$ and $t(-)$.

Table 2

p	$t(+)$	$t(-)$	p	$t(+)$	$t(-)$
7	5	2	47	33	14
11	10	1	53	16	37
13	4	9	59	6	53
17	12	5	61	55	6
19	2	17	67	47	20
23	7	16	71	64	7
29	3	26	73	22	51
31	28	3	79	8	71
37	26	11	83	25	58
41	37	4	89	9	80
43	13	30	97	68	29

(5) The methods so far described have been developed from elementary properties of numbers and are in fact nothing more than extensions of the well-known rules for 'casting out the nines' and detecting multiples of eleven. Constructing algorithms of this sort is always an interesting—and often rewarding—form of mathematical recreation and it is surprisingly easy to devise divisibility tests from purely empirical observations.

For instance the fact that $17 \times 6 = 102$, immediately suggests a simple method of dividing by 17. One has only to double the

left-hand digit, subtract from the next pair of digits and continue in this manner, thus:

$$N = 13579\ (= (17 \times 798) + 13)$$

$$\begin{array}{r} \underline{1}3579 \\ \underline{2} \\ 379 \\ \underline{6} \\ \underline{3}19 \\ \underline{6} \\ 13 = \text{the remainder.} \end{array}$$

Multiplying the underlined digits 133 by 6, (102/17) gives us 798 which is the quotient.

We could, of course, have paired the digits off from the right and proceeded:

$$\begin{array}{ccc} 1 & 35 & 79 \\ & \underline{2} & \\ & 33 & 79 \\ & & \underline{66} \\ & & 13 \end{array}$$

This system is merely long division in a slightly different dialect but it is clearly preferable when dealing with large numbers and divisors which are closely adjacent to powers of 10. Thus one of our earlier 'awkward' divisors, 97, presents no difficulty when it is treated as $(100 - 3)$. Here we use the multiplier 3 and add, as in the example,

$$N = 1234567\ (= (97 \times 12727) + 48)$$

$$\begin{array}{r} \underline{1}234567 \\ \underline{3} \\ \underline{2}64567 \\ \underline{6} \\ \underline{7}0567 \\ \underline{21} \\ \underline{2}667 \\ \underline{6} \\ \underline{7}27 \\ \underline{21} \\ 48 = \text{the remainder} \end{array}$$

And again the underlined digits provide the quotient. Obviously it is a very simple matter to divide by any of the factors of numbers of the forms 9999 . . . 99, and 1000 . . . 01.

I have yet to meet a 'Theory of Numbers' addict who can resist a problem and to round off this chapter I present the following little curiosity. Again use is made of the 't' function, this time in the form of the rising powers $t^0, t^1, t^2, \ldots t^n$, (note that $t^0 = 1$, $t^1 = t$), and the two worked instances should make the principle plain without the need of verbal explanation.

(*A*) $N = 31711 = 19 \times 1669.$ $(t = 2)$

$$\begin{array}{r} 3 \times 1 = 3 \\ 1 \times 2 = 2 \\ 7 \times 4 = 28 \\ 1 \times 8 = 8 \\ 1 \times 16 = \underline{16} \\ 57 \end{array}$$

$$\begin{array}{r} 5 \times 1 = 5 \\ 7 \times 2 = \underline{14} \\ \underline{\underline{19}} \end{array}$$

(*B*) $N = 34561 = 17 \times 2033.$ $(t = 5)$

$$\begin{array}{r} 3 \times 1 = 3 \\ 4 \times 5 = 20 \\ 5 \times 25 = 125 \\ 6 \times 125 = 750 \\ 1 \times 625 = 625 \end{array}$$

Now applying the 'eleven' rule we have,

$$(20 + 750) - (3 + 125 + 625) = 770 - 753 = 17.$$

The point of interest is that the increasing powers of t are applied in the opposite direction (that is, from left to right) to what one might expect.

EXERCISES

1. If $43n = 21010101$, find n by two different calculations, but without using 'long division'.
2. What remainder is left after dividing 110999999 by 997?
3. Devise a simple process for dividing by 167.

3 RECURRING DECIMALS

(1) If we take a large multiple of ten, say 10^n where $n > 20$, and divide it in the normal manner by 19 we produce the sequence of digits 5263157894736842105263 . . . which repeat themselves in bunches of eighteen digits. Dividing this number by the appropriate 10^n we are then able to say that $1/19 = \cdot\dot{0}5263157894736842\dot{1}$, the suffixed periods indicating that this decimal part is repeated over and over again.

There is a simple but important distinction to be observed here. The decimal ·9999 . . ., in which the 9's are continued indefinitely, and usually expressed as $\cdot\dot{9}$, is indistinguishable from unity and the above decimal $\cdot\dot{0}52\ldots842\dot{1}$ is an exact divisor of $\cdot\dot{9}$. On the other hand the number 526 . . . 421 is an exact divisor of 999,999,999,999,999,999. Similarly 142857 is a divisor of 999,999 and hence $\cdot\dot{1}4285\dot{7}$ is a divisor of $\cdot\dot{9}$: it is in fact equal to 1/7.

Reverting again to the fraction 1/19, we saw in the last chapter that division by 19 can be done, in effect, by multiplying successively by 2 starting from the right. It will be seen at once that, starting with the digit 1, the above decimal is an excellent example of this process.

Thus, following this procedure we have:

```
                       168421
                      32
                     64
                   128
                  256
                 512
               1024
              2048
             4096
            8192
          16384
         32768
        65536
      131072
     262144
    524288
  1048576
  etc.
        ----------------------
        4210526315789473684210
```

which, continued indefinitely produces the decimal recurrent cycle of the fraction 1/19. Another way of expressing the above summation is $20^0 + 20^1 + 20^2 + 20^3 + \ldots$, which of course can be summed as follows,

```
          1
         20
        400
       8000
     160000
    3200000
       etc.
    -------
 . . .368421
```

The above sequence of digits can be arrived at in a number of ways. Two of these will be illustrated next in order to indicate the diversity of approach which this subject provides.

The following tabulation will probably demonstrate one method more clearly than any laboured description of the process.

Three columns, A, B, and C are set up in which B_n is the

integral part of $A_n/2$ and C_n is the remainder. The A_n entries, starting with 1, are found from the relationship

$$A_{n+1} = 10C_n + B_n$$

	A	B	C
2 into	1	= 0 remainder	1
	10	5	0
	5	2	1
	12	6	0
	6	3	0
	3	1	1
	11	5	1
	15	7	1
	17	8	1
	18	9	0
	9	4	1
	14	7	0
	7	3	1
	13	6	1
	16	8	0
	8	4	0
	4	2	0
	2	1	0
	1	0	1

Column B then provides the required sequence of digits.

The next example is again best expressed in columnar form. This time the successive A entries are multiplied by 10 and we have the relationship $A_{n+1} = 10C_n - B$, where B is the largest multiple of 19 less than A_{n+1}.

A	B	C
$10 \times 1 -$	$0 =$	10
$10 \times 10 -$	$95 =$	5
$10 \times 5 -$	$38 =$	12
$10 \times 12 -$	$114 =$	6
$10 \times 6 -$	$57 =$	3
$10 \times 3 -$	$19 =$	11
$10 \times 11 -$	$95 =$	15
$10 \times 15 -$	$133 =$	17

and so on.

The curiosity here is that the 'units' digits of column C are now those of column B in the last example whilst the 'tens' digits (which are to be ignored for the present purpose) occur in exactly the same positions as the remainders '1' appeared in the previous column C.

I am sure readers would find it entertaining to find for themselves the corresponding techniques for determining the recurring cycles of the fraction 1/29. There are, as one might expect, close relationships amongst all fractions of the form $1/(10n \pm 1)$ and it is not difficult, reasoning by analogy, to formulate general constructions.

(2) These periodic sequences of digits relating to the common fractions have many curious properties, not all of them as predictable as one might expect and in order to observe some of them and to provide enough material for further experiment it will be useful to have before us a sufficient number of examples. Table 3 lists the decimal equivalents of the reciprocals of the prime numbers from 7 to 97 inclusive.

Table 3

Recurrent Period of $1/p$. (p—prime)

p	
7	·142857
11	·09
13	·076923
17	·0588235294117647
19	·052631578947368421
23	·0434782608695652173913
29	·0344827586206896551724137931
31	·032258064516129
37	·027
41	·02439
43	·023255813953488372093
47	·0212765957446808510638297872340425531914893617
53	·0188679245283
59	·0169491525423728813559322033898305084745762711864406 779661
61	·0163934426229508196721311475409836065573770491803278 68852459
67	·014925373134328358208955223880597
71	·01408450704225352112676056338028169
73	·01369863
79	·0126582278481
83	·01204819277108433734939759036144578313253
89	·01123595505617977528089887640449438202247191
97	·0103092783505154639175257731958762886597938144329896 907216494345360824742268041237113402061855 67

It will be seen at once that roughly half of these periods contain $(p - 1)$ digits whilst the rest have some fraction of $(p - 1)$. When the denominator of the original common factor is composite (say m) the number of digits in the period is, however, invariably less than m. For further reference and to provide material for any reader who wishes to pursue the subject, Tables 4 and 5 give the decimal equivalents of the reciprocals of some composite numbers.

Table 4

Recurrent period of $1/m$. (m—composite, $\neq 2k$, $5k$, or k^2)

m	
21	·047619
27	·037
33	·03
39	·025641
51	·0196078431372549
57	·017543859649122807
63	·015873
69	·0144927536231884057971
77	·012987
87	·0114942528735632183908045977
91	·010989
93	·010752688172043

Table 5

Recurrent period of $1/p^2$. ($p \neq 2k$ or $5k$)

p^2	
49	·020408163265306122448979591836734693877551
81	·012345679
121	·0082644628099173553719
169	·0059171597633136094674556213017751479289940828402366 863905325443786982248520 71
289	·0034602076124567474048442906574394463667820069204152 2491349480968858131487889273356401384083044982698969 1937716262975778546712802768166089965397923875432525 9 5155709342560553633217993079584775086505190311418685 1211072664359861591695501730103806228373702422145328 719723183391
361	·0027700831024930747922437673130193905817174515235457 0637119113573407202216066481994459833795013850415512 4653739612188365650969529085872576177285318559556786 7036011080332409972299168975069252077562326869806094 1828254847645429362880886426592797783933518005540166 2049861495844875346260387811634349030470914127423822 714681440443213296398891966759
441	·002267573696145124716553287981859410430839

(3) With these tables we now have enough material to make a number of observations of the properties and methods of construction of these interesting periodic sequences.

As we saw at the beginning of this chapter the fraction 1/19 is generated by multiplying successively by the '*t*' function (see Table 2) from the right and starting with unity. This method is quite generally applicable and is not restricted to prime denominators. For instance the decimal equivalent of 1/39 as seen in Table 3 is $0{\cdot}\dot{0}2564\dot{1}$, obtained by using the multiplier 4.

Thus:

```
              1
             4
           16
          64
        256
      1024
    4096          etc.
----------------
. . . 1025641
```

Multiplying this sequence by three gives . . . 3076923 . . . (1/13). Of course the periodic part of the fraction 1/13 could have been obtained directly by multiplying as before by four but starting with the digit 3, thus:

```
              3
            12
           48
         192
        768
      3042
    12288         etc.
----------------
. . . 3076923
```

Similarly, using $\times 5$, and starting with 7 we obtain the sequence . . . 142857 for the fraction 1/7.

Using the above principles together with the '*t*' functions in Table 2, we have then a general method for producing the periodic parts of any fraction whose denominator is not a multiple of two or five.

There are many other ways of doing this. As an example of one employing a more orthodox arithmetic than the above we may note that on dividing seven into one hundred a remainder of two

is left; the following two hundred then has a remainder of four, and so on. The decimal equivalent of 1/7 can then be quickly written down by multiplying the first dividend by two, placing the result two places to the right and continuing in this way, thus:

```
14
  28
    56
     112
       224
         448     and so on.
------------
1428571428 . . .
```

This example was chosen because it is often quoted as a curiosity; in fact since $10 - 7 = 3$ it is just as effective to use 3/10 as the multiplier.

```
1
 3
  9
  27
   81
   243
    729
    2187
     6561
     19683
      59049
-----------
1428571 . . .
```

Being a straightforward substitute for long division there are no restrictions in this process and for example the fraction 1/8 can be calculated in an exactly similar manner. Here $8 = 10 - 2$ and we have,

```
        1
         2
          4
          8
         16
          32
           64
          128
           256 . . .
-------------------
. . . 1249999 . . .    = ·125
```

The method is obviously quite general, its main charm as a computational time-saver arising when the denominator of the required fraction or one of its multiples is close to and a little less than a power of ten. For instance, noting that $17 \times 588 = 9996 = 10000 - 4$, we can rapidly decimalise $1/17$ by multiplying $\cdot 0588$ successively by $4/10000$, thus:

$$\begin{array}{l}
0588 \\
2352 \\
9408 \\
37632 \\
150528 \\
602112\ldots \\
\hline
05882352941176470588\ldots
\end{array}$$

The above examples lead back in a roundabout way to the second paragraph of this chapter, for clearly if a fraction's denominator is a factor of $10^n - 1$, then the multiplier is unity. In such cases it is easy now to see that since, say, $999 = 37 \times 27$, then $1/37 = \cdot 0270270\ldots = \cdot\dot{0}2\dot{7}$, and as $99999 = 41 \times 2439$, then $1/41 = \cdot\dot{0}243\dot{9}$.

The situation changes in a curious and less obvious manner when a denominator, or one of its multiples, is treated as exceeding a

power of ten. Taking again $1/17$ as an example we have $17 \times 6 = 102$, and we now use the multiplier $2/100$ but this time alternately add and subtract the successive multiples. Thus:

$$\begin{array}{ll}
+ & \cdot 06 \\
- & 12 \\
\hline
 & \cdot 0588 \\
+ & 24 \\
\hline
 & \cdot 058824 \\
- & 48 \\
\hline
 & \cdot 05882352 \\
+ & 96 \\
\hline
 & \cdot 0588235296 \\
- & 192 \\
\hline
 & \cdot 058823529408 \\
+ & 384 \\
\hline
 & \cdot 05882352941184 \\
- & 768 \\
\hline
 & \cdot 0588235294117632 \\
+ & 1536 \\
\hline
 & \cdot 058823529411764736 \\
- & 3072 \\
\hline
 & \cdot 058823529411764705\ldots
\end{array}$$

By a similar process, and noting that $7 \times 143 = 1001$, we have a still quicker method for determining $1/7$, namely,

$$\begin{array}{ll}
+ & \cdot 143 \\
- & 143 \\
\hline
 & \cdot 142857 \\
+ & 143 \\
\hline
 & \cdot 14285714\ldots
\end{array}$$

(4) It is in fact amazing how many algorithms can be devised for constructing the recurring periods of decimal fractions. Added to what has already been shown the following three examples will indicate something of the richness of this field.

Example 1. 1/31

$$\begin{array}{lcr}
 & & 29 \\
2 \times 29 + 3 & = & 61 \\
2 \times 61 + 29 & = & 151 \\
2 \times 151 + 61 & = & 363 \\
2 \times 363 + 151 & = & 877 \\
2 \times 877 + 363 & = & 2117 \\
2 \times 2117 + 877 & = & 5111 \\
2 \times 5111 + 2117 & = & 12339 \\
2 \times 12339 + 5111 & = & 29789 \\
\hline
 & & \ldots 29032258064516129
\end{array}$$

Example 2. 1/29

$$\begin{array}{lcr}
 & & 31 \\
9 \times 31 & = & 279 \\
9 \times 279 & = & 2511 \\
9 \times 2511 & = & 22599 \\
9 \times 22599 & = & 203391 \\
9 \times 203391 & = & 1830519 \\
\hline
 & & \ldots 551724137931
\end{array}$$

Example 3. 1/67

$$\begin{array}{lcl}
 & & \cdot 01 \\
 & & \;\;49 \\
(1 + 49)/2 & = & \;\;\;\;25 \\
(49 + 25)/2 & = & \;\;\;\;\;\;37 \\
(25 + 37)/2 & = & \;\;\;\;\;\;\;\;31 \\
(37 + 31)/2 & = & \;\;\;\;\;\;\;\;\;\;34 \ldots \\
\hline
 & & \cdot 014925373134 \ldots
\end{array}$$

(5) There are some still more remarkable ways in which these periodic sequences can be formed, illustrating in a striking manner how an unsuspected common link can sometimes be found between apparently unrelated branches of Number Theory. It will be more convenient, however, to examine these after introducing in the next chapter the curious additive series associated with the name of Fibonacci.

From the constructions already given it will have become clear that fractions which have recurrent, or cyclic, periods can also be

expressed by the summation of infinite series. We have seen, in effect, that $1/7$ $(= 0{\cdot}\dot{1}4285\dot{7})$ is equivalent to

$$\frac{1}{10} + \frac{3}{10^2} + \frac{3^2}{10^3} + \frac{3^3}{10^4} + \ldots$$

and also to

$$\frac{14}{10^2} + \frac{28}{10^4} + \frac{56}{10^6} + \frac{112}{10^8} \ldots$$

which is the same as

$$\frac{7}{50} + \frac{7}{50^2} + \frac{7}{50^3} + \ldots$$

and again, to

$$\frac{143}{10^3} - \frac{143}{10^6} + \frac{143}{10^9} - \ldots$$

It might also be noted that the series for $1/19$ and $1/21$ can be simplified into the following forms:

$$1/19 = 1/20 + 1/20^2 + 1/20^3 + \ldots$$
$$1/21 = 1/20 - 1/20^2 + 1/20^3 - \ldots$$

In general, if we express N as one of the forms $(10^n \pm a)/s$ the series for $1/N$ then becomes:

(1) $$\frac{1}{N} = \frac{s}{10^n - a} = \frac{sa}{10^n} + \frac{sa^2}{10^{2n}} + \frac{sa^3}{10^{3n}} + \ldots$$

(2) $$\frac{1}{N} = \frac{s}{10^n + a} = \frac{sa}{10^n} - \frac{sa^2}{10^{2n}} + \frac{sa^3}{10^{3n}} - \ldots$$

Some of the cyclic properties of recurrent decimals are of common knowledge, particularly the confined cycle which occurs when $1/N$ has a period consisting of $N - 1$ digits. With a denominator of seven, for instance, the corresponding six fractions are:

$$\begin{array}{l}
1/7 = 0{\cdot}142857 \ldots \\
2/7 = 0{\cdot}285714 \ldots \\
3/7 = 0{\cdot}428571 \ldots \\
4/7 = 0{\cdot}571428 \ldots \\
5/7 = 0{\cdot}714285 \ldots \\
6/7 = 0{\cdot}857142 \ldots
\end{array}$$

This recycling of the same digits in the same order is to be found in the fractions 1/17, 1/19, 1/23, 1/29, 1/47, 1/59, 1/61, 1/67, 1/97, . . ., but it obviously cannot occur when the number of digits in the cycle is less than $N - 1$. Thus when N is 13 and 1/13 has the decimal equivalent of ·076923 employing only $(N - 1)/2$ digits, we have two sets of recycling digits.

1/13 = ·076923 . . .,	2/13 = ·153846 . . .
3/13 = ·230769 . . .,	5/13 = ·384615 . . .
4/13 = ·307692 . . .,	6/13 = ·461538 . . .
9/13 = ·692307 . . .,	7/13 = ·538461 . . .
10/13 = ·769230 . . .,	8/13 = ·615384 . . .
12/13 = ·923076 . . .,	11/13 = ·846153 . . .

The two sets of numerators, (1, 3, 4, 9, 10, 12) and (2, 5, 6, 7, 8, 11), have interesting derivations and properties. For instance, if a list is made of the squares of the natural numbers 1, 2, 3, . . ., and from these we subtract the largest multiple of thirteen that will leave a positive remainder, we have:

1	4	9	16	25	36	49	64	81	. . .
			13	13	26	39	52	78	. . .
1	4	9	3	12	10	10	12	3	. . .

which provides a recurrent cycle (in a different order) of the numbers 1, 3, 4, 9, 10, 12, that is, the first set of numerators shown above.

Again, the two sets, which we will call respectively A and B, have a curious internal property. Multiplying any two or more of the numbers of set A together and then subtracting a suitable multiple of thirteen leaves a remainder which is one of the numbers in the set. Thus $(3 \times 12) - 26 = 10$, and $(3 \times 4 \times 9) - (8 \times 13) = 4$, and so on. Moreover, on applying the same procedure to members of set B we again arrive at the numbers of set A, as follows $(2 \times 8) - 13 = 3$. On the other hand if a number from set A is multiplied by one from set B the outcome is a member of set B.

It would be impracticable to pursue this line of thought further until some experience of congruences and quadratic residues has been gained but it must occasion some surprise that there is a direct link between recurrent decimals and the square numbers.

(6) And finally to round off this chapter I am going to revert to a construction method which the reader might care to experiment with and/or find an explanation.

We see from Table 5 that 1/121 is equal to

$$\cdot\dot{0}08264462809917355371\dot{9}\ .\ .\ ..$$

The following operations require no verbal description.

8	= 8
8 + 18 =	26
26 + 18 =	44
44 + 18 =	62
62 + 18 =	80
80 + 18 =	98
98 + 18 =	116
116 + 18 =	134
134 + 18 =	152
152 + 18 =	170
170 + 18 =	188
188 + 18 =	206
206 + 18 =	224 etc.
	826446280991735537190082 . . .

1 + 18 =	19
19 + 18 =	37
37 + 18 =	55
55 + 18 =	73
73 + 18 =	91
91 + 18 =	109
109 + 18 =	127
127 + 18 =	145
145 + 18 =	163
163 + 18 =	181
181 + 18 =	199 etc.
	. . . 0082644628099173553719

EXERCISES

1. What are the decimal equivalents of

$$3/41,\ 7/41,\ 13/41,\ 29/41,\ \text{and}\ 30/41?$$

2. Find empirically some algorithm which generates the recurrent period of the fraction 1/441. ($= 1/21^2$), namely

·00226757369614512471655328798185941043083 9 . . .

This exercise might be expanded into a general search for algorithms among these periods. Anyone with a flair for empirical observation will find ample scope for ingenuity in this field.

4 ADDITIVE SERIES

(1) One of the most exciting things about experimenting with numbers is that one can so rapidly find oneself in strange and unfamiliar surroundings. And in such a situation there is little that can compete with the thrill of coming across some unexpected link with an apparently unrelated branch of mathematics.

A particular and most interesting example of this is to be found in a study of the additive series originated by Leonardo de Pisa, nowadays better known as Fibonacci. This series, first defined in the early part of the thirteenth century, is usually presented in the form 0, 1, 1, 2, 3, 5, 8, 13, . . ., each new term being the sum of the two preceding terms. In actual fact the fundamental properties of this important series would remain unchanged whatever integers were used to start it. Thus, observing the rule and giving any value we like to a, and b, the series becomes:

a, b, $(a + b)$, $(a + 2b)$, $(2a + 3b)$, $(3a + 5b)$, $(5a + 8b)$, etc.

For reasons which will be clarified later it will be found convenient to re-write the series in the form of two columns A and B, the respective terms of which are determined by the following instructions:

$$A_{n+1} = A_n + B_n,$$
$$B_{n+1} = B_n + A_{n+1}.$$

Taking successive values from the above table and determining the decimal equivalents of the fractions B_n/A_n and $(A_{n+1})/B_n$ it will be found that these both converge towards the value

$$1{\cdot}6180339888\ldots,$$

thus,

$$\begin{array}{ll} 89/55 = 1{\cdot}61818\ldots, & 144/89 = 1{\cdot}61765\ldots \\ 233/144 = 1{\cdot}61806\ldots, & 377/233 = 1{\cdot}61798\ldots \\ 610/377 = 1{\cdot}61804\ldots, & 987/610 = 1{\cdot}61803\ldots \end{array}$$

and so on.

Table 6

The Reconstructed Fibonacci Series

n	A	B	
1	0	1	
2	1	2	
3	3	5	
4	8	13	
5	21	34	
6	55	89	
7	144	233	
8	377	610	
9	987	1597	
10	2584	4181	etc.

We come now to a remarkable property of this series. The fractions A_n/B_n and $B_n/(A_{n+1})$ get closer and closer, as n increases, to 0·61803 . . . (for example $89/144 = 0{\cdot}61805\ldots$), while the fractions $(A_{n+1})/A_n$ and $(B_{n+1})/B_n$ approach 2·61803 . . ..

The Greeks knew this proportion (1·61803 . . .) as the Golden Ratio and made frequent use of it in their architecture and sculpture. It seems to have a curiously aesthetic attraction and many famous canvasses have dimensions closely approximating to adjacent members of the series. The ratio is now more commonly referred to as the Golden Section and is variously indicated by the Greek symbols for 'phi' or 'tau'.

It is the ratio of a diagonal to a side in a regular pentagon and of the radius of a circle to the side of an inscribed regular decagon. It forms the ratio between the number of clockwise and anti-clockwise spirals in the seed patterns of such flowers as sunflowers and in those of the areoles of globular cacti. It can be used to generate the logarithmic spiral governing the growth of sea and snail shells and the distribution of leaves around the stems of many plants, and at least one dedicated research worker claims to have correlated it to the ratio of the height of women's navels to their total stature.

In spite, or perhaps because, of the richly coloured applications that have been claimed for this innocent looking series it has come to be regarded more as a mathematical curiosity than as a subject for serious study and has received scant attention in most writings on the higher arithmetic. In what follows I hope to modify this

impression and to show that there is a sound mathematical content in series of this form.

(2) First to deal with the fairly well-known properties of the Golden Ratio. Partly to avoid conflict with the adherents of 'phi' and 'tau', and partly to cope with the limitations of my typewriter, I shall use 'F' to indicate 1·61803 . . ..

It is readily verified that $(1{\cdot}618\ldots)^2 = 2{\cdot}618\ldots$, and that $1/1{\cdot}618\ldots = 0{\cdot}618\ldots$. Thus F is a number whose reciprocal is obtained by subtracting unity and which is squared by adding unity. Both of these statements, $1/F = (F - 1)$ and $F^2 = F + 1$ produce the quadratic equation

$$F^2 - F - 1 = 0$$

Solving this by normal algebraic procedure we obtain:

$$F = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a} = (1 + \sqrt{5})/2 = 1{\cdot}618\ldots$$

It follows that $1/F = (\sqrt{5} - 1)/2 = 0{\cdot}618\ldots$, and since $\sqrt{5}$ is irrational, that is its decimal part neither terminates nor recurs, then F is also irrational.

Both F and the series from which it is derived have strange properties. For instance, keeping one eye on Table 6, note the following relationships:

$$\begin{aligned} 2 \times 5 - 3^2 &= 2^2 - 1 \times 3 = 1 \\ 5 \times 13 - 8^2 &= 5^2 - 3 \times 8 = 1 \\ 13 \times 34 - 21^2 &= 13^2 - 8 \times 21 = 1 \\ 34 \times 89 - 55^2 &= 34^2 - 21 \times 55 = 1 \qquad \text{and so on.} \end{aligned}$$

The reader may well find it interesting to search for other associations, perhaps using different starting numbers for 'A' and 'B', but I think it is safe to predict that he will find it difficult wholly to escape from the sequential integers of the series.

Even the ascending powers of F cannot disengage themselves as will be seen from the following relationships,

$$\begin{aligned} F &= F \\ F^2 &= F + 1 \\ F^3 &= 2F + 1 \\ F^4 &= 3F + 2 \\ F^5 &= 5F + 3 \\ F^6 &= 8F + 5 \\ F^7 &= 13F + 8 \qquad \text{and so on.} \end{aligned}$$

But perhaps the most unexpected property of this series emerges when the two numbers 1·61803 . . . and 2·61803 . . . are multiplied in turn by the natural numbers 1, 2, 3, 4, . . ., the fractional parts of the products being ignored. The first few pairs of numbers formed in this way are:

(1, 2), (3, 5), (4, 7), (6, 10), (8, 13), (9, 15), (11, 18),

and it can be shown that however far the operation is carried no integer is repeated and none omitted.

(3) References to the 'magical' properties of the Golden Section are often to be found in articles dealing with mathematical puzzles and paradoxes and it is often claimed, or at least implied, that it occupies an unique position in regard to the peculiarities which have just been mentioned. This may be true in some senses but let us return to our presentation of the Fibonacci series in two columns and try a slight change in its formulation.

The 'A' and 'B' columns will now be constructed according to the rules:

$$A_{n+1} = A_n + B_n$$
$$B_{n+1} = A_n + A_{n+1}$$

We then have,

Table 7

n	A	B
1	0	1
2	1	1
3	2	3
4	5	7
5	12	17
6	29	41
7	70	99
8	169	239
9	408	577
10	985	1393
11	2378	3363
12	5741	8119
13	13860	19601
. . .	. . .	. . .

One of the advantages of using a two column presentation of such series will now become apparent; stretched out in a single line as Fibonacci's series is usually written it would be difficult to recognise this one as purely additive. However by its method of construction it most certainly is, and from an arithmetical point of view it is equally interesting.

Applying to the values in Table 7, the technique which provided the F number 1·61803 . . . and its derivatives we find that the ratio B_n/A_n approaches, as n increases, the limit 1·41421356237 . . ., which is of course the decimal expression of $\sqrt{2}$.

Furthermore the ratios $(A_{n+1})/A_n$ and $(B_{n+1})/B_n$, as n increases, approach the value $\sqrt{2} + 1$, $(B_{n+1})/A_n$ gets nearer and nearer to $\sqrt{2} + 2$, and so on. Incidentally, these ratios and the general properties of this series are, as in the Fibonacci construction, quite unaffected by the choice of the two starting numbers.

This latter series also shares what seemed to be a property peculiar to the 'F' numbers. Multiplying the values 1·4142 . . . and 3·4142 . . . successively by the whole numbers 1, 2, 3, 4, . . . and discarding the fractional parts of the products we again find a sequence of pairs of numbers which includes all the natural numbers without duplication. The first few of these pairs are:

(1, 3), (2, 6), (4, 10), (5, 13), (7, 17), (8, 20), (9, 23), . . .

and here we see that the differences between the numbers in brackets are successively 2, 4, 6, 8, etc.

(4) Striking as these properties and the relationships between the two series are, there is a still more remarkable connection to be noticed between them and some of the periods of the recurrent decimals examined in Chap. 3. Four examples from my own observations will be given and I have no doubt the reader will find this an exciting field for further research.

(A) If we take the original Fibonacci series, 0, 1, 1, 2, 3, 5, etc., and divide each term in turn by 10, 100, 1000, and so on, and then sum the results we have:

·0112358
13
21
34
55
89
144
233
377
610
987
1597
. . .

·011235955056179 . . . = 1/89 (see Table 3).

(*B*) If, on the other hand, the alternate terms of this series—as shown in column *A* of Table 6—are treated in the above manner we arrive at the following summation:

·0138
21
55
144
377
987
2584
6765
17711
46368
121393
317811
832040
2178309
5702887
14930352
39088169
102334155
. . .

·014084507042 . . . = 1/71.

(*C*) Now we stated earlier—and this is easily demonstrated—that the ratio between successive terms of the Fibonacci series approaches

F whatever integers are used to start the sequence. Again using the above process but this time starting with the numbers 5, and 26, and dividing the successive terms by 10^2, 10^4, 10^6 etc., we now have:

·0526315788
145
233
378
611
989
1600
2589
4189
. . .

·05263157894736842105263 . . . = 1/19

(*D*) Turning now to (4, 3), in which a series producing the ratio 1·41421 . . . ($\sqrt{2}$) was described, it will be seen that this sequence of numbers also has its surprises. Taking alternate terms—that is the consecutive lines of column *A* in Table 7—and treating them as in (*A*) and (*B*) above we obtain the decimal equivalent of the fraction 1/79, thus:

·0125
12
29
70
169
408
985
2378
5741
13860 and continuing,
33461
80782
195025
470832
1136689
2744210
. . .

·01265822784810 . . . = 1/79.

I think this link with the recurrent periods of 'prime denominator' fractions presents additive series in a new light and certainly there is scope here for some research.

EXERCISES

1. Calculate $6F^2/5$ to five significant figures.

$$(F = 1{\cdot}61803\ldots)$$

2. What is the sum of the first n Fibonacci numbers?

5 ONE ONE ONE ...... ONE

(1) In the previous chapters we have frequently run up against those numbers which belong to one or the other of the forms $10^n - 1$ or $10^n + 1$. Expressed in digital language these are respectively of the general types

$$9999\ldots99 \text{ and } 1000\ldots01.$$

We have seen that rapid checks of divisibility can easily be applied to the factors of such numbers and that they have an important bearing upon the structure of recurring decimals; for these and a variety of other reasons it will be of interest to examine them in greater detail and to discover some of their special properties and behaviour.

All integers of the form $A^n - 1$ are divisible by $A - 1$ and therefore in dealing with numbers of the form $10^n - 1$ it will be most convenient first to remove the factor 9 and concentrate upon the sequence 1, 11, 111, . . ., 111 . . . 1. In what follows such integers will be referred to as 'I' numbers, and in particular I_n will be used to indicate $(10^n - 1)/9$, or the integer 111 . . . 1 having n digits. Thus, for example, $I_5 = 11111 = (10^5 - 1)/9$.

It will also be convenient to use a similar convention to describe integers of the form $10^n + 1$; these will be referred to as 'J' numbers, the subscript again being equal to n. Thus J_5, or $10^5 + 1$ represents 100001, n in this case being the number of zeros plus one.

By definition $I_n \times J_n = I_{2n}$ and in general it is easily seen that when n is composite I_n can always be resolved into at least two factors, as for example:

$$\begin{aligned} I_{12} &= 11 \times 10101010101 = 111 \times 1001001001 \\ &= 1111 \times 100010001 = 111111 \times 1000001 \end{aligned}$$

I_n can therefore only be prime when n is prime and indeed it is one of the most remarkable properties of both the 'I' and 'J'

numbers that they seem to be extraordinarily free from primes. (I say 'seem' because the existing tests of primality become exceedingly difficult to apply when n exceeds 37.)

I_2 (= 11) is certainly prime, I_{19} is generally accepted to be prime, although I have not found any reference to a *positive* proof of this. On the other hand I_{23} is still widely quoted as being prime in spite of a somewhat dubious ancestry and a proof of its composite character by D. H. Lehmer in 1927. Beyond this there are no more prime I_n's up to, at least $n = 137$.

(2) One might expect that numbers constructed in such an orderly and regular fashion would exhibit some sort of pattern in the distribution of their factors. As we shall see later this is to a certain extent true but at first sight complete factorisation presents a chaotic picture.

The following Tables 8 and 9, list the factors of the smaller I and J numbers. I think these factors are all prime but am not completely certain about the large factors of I_{25}, and those of J_n where $n = 20, 22$, and 23. Perhaps after studying the factorisation methods which will be described later some readers may find the subject interesting enough to try their hand at cracking these somewhat formidable numbers.

Table 8

Factors of $I_n \left(= \dfrac{10^n - 1}{9}\right)$

n	*Factors*
2	11 (prime)
3	3.37
4	11.101
5	41.271
6	3.7.11.13.37
7	239.4649
8	11.73.101.137
9	3.3.37.333667
10	11.41.271.9091
11	21649.513239
12	3.7.11.13.37.101.9901
13	53.79.265371653
14	11.239.4649.909091
15	3.31.37.41.271.2906161
16	11.17.73.101.137.5882353
17	2071723.5363222357
18	3.3.7.11.13.19.37.52579.333667
19	1111111111111111111 (thought to be prime)
20	11.41.101.271.9091.99009901
21	3.37.43.239.1933.4649.10838689
22	11.11.23.4093.8779.21649.513239
23	(Composite but factors not known)
24	3.7.11.13.37.73.101.137.9901.99990001
25	41.271.1000010000100001000001
26	11.53.79.859.265371653.1058313049
27	3.3.3.37.757.333667.440334654777631
28	11.29.101.239.281.4649.909091.121499449
29	3191.16763.43037.62003.77843839397
30	3.7.11.13.31.37.41.211.241.271.2161.9091.2906161
32	11.17.73.101.137.353.449.641.1409.69857.5882353
34	11.103.4013.2071723.5363222357.21993833369
36	3.3.7.11.13.19.37.101.9901.52579.333667.999999000001
38	11.1111111111111111111.909090909090909091

Incidentally, I_{38} is unique among the numbers listed above in that it is the only one, with n even, that has *three* prime factors.

Table 9

Factors of J_n (= $10^n + 1$)

n	*Factors*
1	11 (prime)
2	101 (prime)
3	7.11.13
4	73.137
5	11.9091
6	101.9901
7	11.909091
8	17.5882353
9	7.11.13.19.52579
10	101.99009901
11	11.23.4093.8779
12	73.137.99990001
13	11.859.1058313049
14	29.101.281.121499449
15	7.11.13.211.241.2161.9091
16	353.449.641.1409.69857
17	11.103.4013.21993833369
18	101.9901.999999000001
19	11.909090909090909091
20	73.137.9999000099990001
21	7.7.11.13.127.2689.459691.909091
22	89.101.1112470797641561909
23	11.47.139.2531.549797184491917
24	17.5882353.9999999900000001
25	11.251.5051.9091.78875943472201

(3) Of course, if we are content to express these numbers by, at most, only two factors a much more regular pattern emerges, as exemplified in the following breakdown of the J numbers.

J_2	101	J_3	11.91
J_4	10001	J_5	11.9091
J_6	101.9901	J_7	11.909091
J_8	100000001	J_9	11.90909091
J_{10}	101.99009901		etc.

In fact these numbers containing only the digits 0, 1, and 9, arranged in various patterns offer in themselves an interesting subject for research; one set in particular provides an excellent example of the dangers of drawing hasty conclusions in enquiries of this kind. Taking in sequence one of the factors of J_{3n} (or I_{6n}) we have:

J_3	91	composite
J_6	9901	prime
J_9	999001	composite
etc.	99990001	prime
	9999900001	composite
	999999000001	prime
	99999990000001	composite
	9999999900000001	prime
	999999999000000001	composite

One might be tempted to infer that the next number in the sequence (99999999990000000001) is prime but fortunately as it must be a factor of I_{60} it is not difficult to prove that it is composite.

Since 60 contains the factors 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, it follows that I_{60} must contain all the factors appearing in the numbers $I_2, I_3, I_4, \ldots I_{30}$, namely, (11), (3.37), (101), (41.271), (7), (9091), (9901), (31.2906161), (99009901), and (211.241.2161). It is easy to see that if these factors are multiplied together the product must have more than forty digits and hence I_{60} cannot have a single prime factor of twenty digits.

The number 99999999990000000001 must therefore have at least one of the above primes as a factor.

(4) It will be seen from Tables 8 and 9, or for that matter readily deduced from first principles, that divisors of I_n numbers are also divisors of I_{mn} where m is one of the natural numbers 1, 2, 3, ... Consequently when dealing with the I and J numbers we need really only be concerned with those primes which appear as divisors *for the first time*. These will recur in all multiples of n and further research on numbers of this form will be made easier by abstracting the primary factors and presenting them in tabular form as in the following Table 10.

Table 10

Primary Factors of I_n

n	*Factors*
2	11
3	3.37
4	101
5	41.271
6	7.13
7	239.4649
8	73.137
9	333667
10	9091
11	21649.513239
12	9901
13	53.79.265371653
14	909091
15	31.2906161
16	17.5882353
17	2071723.5363222357
18	19.52579
19	1111111111111111111
20	99009901
21	43.1933.10838689
22	23.4093.8779
23	not known
24	99990001
25	1000010000100001000010001
26	859.1058313049
27	757.440334654777631
28	29.281.121499449
29	3191.16763.43037.62003.77843839397
30	211.241.2161
31	2791.*N*
32	353.449.641.1409.69857
33	67.*N*
34	103.4013.21993833369
35	71.*N*
36	999999000001
37	2028119.*N*

This table provides a number of working examples of one of the most important theorems in the Theory of Numbers and at the same time an insight into a puzzling feature of the varying periods of recurring decimals.

In the first place, remembering that factors of I_n are also factors of I_{nm} we see that:

7 is a factor of I_6
11 is a factor of I_{10}
13 is a factor of I_{12}
17 is a factor of I_{16}

and so on.

And generally, when n is prime it is a divisor of I_{n-1}.

Bearing in mind how the I_n numbers are constructed this is seen to be a particular example of 'Fermat's Theorem' which states that if p is prime and N is not a multiple of p, then $N^{p-1} - 1$ is divisible by p exactly. This theorem is of fundamental utility in the exploration of numbers; proofs, depending usually either on binomial expansion or congruence theory, will not be given here as they are to be found in almost every textbook on Algebra.

The table also serves to illustrate the following lesser—but still important—theorems. Here again the proofs, which though not difficult depend upon a sequence of lemmas, are left to the formal textbooks. (For instance, *Advanced Algebra*, Barnard and Child Macmillan & Co. Ltd.)

(1) If n is an odd prime, every prime factor of $N^n - 1$ which is not a divisor of $N - 1$ is of the form $2kn + 1$.

(2) If p is a prime factor of $N^n + 1$ it is also a factor of $N^d + 1$ where d is the greatest common divisor of n and $\frac{1}{2}(p - 1)$.

(3) If n is a prime, every odd prime factor of $N^n + 1$ which is not a divisor of $N + 1$ is of the form $2kn + 1$.

(4) If n is a prime, every prime factor of $N^{n^x} - 1$ which is not a divisor of $N^{n^{x-1}} - 1$ is of the form $kn^x + 1$.

(5) Every prime factor of $N^{2^n} + 1$ is of the form $2^{n+1}.\,k + 1$.

These theorems are of value in limiting the choice of possible factors of both I and J numbers and as we shall see later it is possible to narrow the field still further.

(5) Before pursuing this line however, it will be of interest to note a connection between recurring decimals and Table 10. Referring back to Table 3 and its following paragraph we recall that the recurrent period of the decimal form of $1/p$ contains a number of digits which is equal either to $p - 1$ or to some divisor of $p - 1$. Thus $1/7$, $(= \cdot\dot{1}4285\dot{7})$ has six digits whilst $1/53$, $(= \cdot\dot{0}18867924528\dot{3})$ has $52/4 = 13$ digits. At first sight the period lengths appear to be decided in an entirely random manner.

But the factors of I_n are in effect the factors of 999 . . . 9 (having n 9's) and the periods of recurring decimals are exact divisors of numbers of this kind. In consequence the reciprocals of the *factors* shown in Table 10 have the corresponding n number of digits in their recurring period. Thus both $1/41$ $(= \cdot\dot{0}243\dot{9})$, and $1/271$ $(= \cdot\dot{0}036\dot{9})$ have five-digit cycles, $1/7$, and $1/13$, six-digits, and so on. Furthermore in the whole infinity of the natural numbers there can only be one whose reciprocal has exactly 2, 4, 10, . . . 19, . . . 36, etc., digits.

(6) The factorisation of I numbers presents a challenge which is difficult to resist and although we shall deal with general methods of factorisation later this is an appropriate point to mention some aids which can be applied to these numbers in particular. Unfortunately it is outside the scope of this book to discuss the theory of Quadratic Residues (that is, the remainders left after subtracting a given prime or its multiples from the sequence of square numbers) but one of its consequences must be mentioned in this context.

Numbers which have a Quadratic Residue of 10 are of the general form $40k + 1, 3, 9, 13, 27, 31, 37, 39$, whilst those with a Q.R. of -10 are of the form $40k + 7, 11, 17, 19, 21, 23, 29, 33$. It can be shown that primary factors of I_n (i.e. n odd) must be of the first form whilst those of J_n are of the second. This, I am afraid, will have to be taken on trust as far as formal proof is concerned but the factors given in Table 10 and a subsequent table (12) will show a practical confirmation of these statements.

The important thing is that we now have two independent 'forms' of factors of these numbers and these can be combined to give a much more economical approach to their factorisation.

To take a specific example let us examine the possible factors of I_7 (= 1,111,111). These must be of one or more of the forms $40s + 1, 3, 9, 13,$ etc., and at the same time of the form $14t + 1$: equating

these forms in turn we have:

(A) $14t + 1 = 40s + 1$, and hence $7t = 20s$. This equation is satisfied when $t = 20, 40, . . . 20k$. Substituting, we have $14(20k) + 1 = 280k + 1$.

(B) $14t + 1 = 40s + 3$, thus $7t = 20s + 1$, which is satisfied when $t = 3, 23, 43, . . . 20k + 3$. Then $14(20k + 3) + 1 = 280k + 43$.

Continuing in this way it is not difficult to determine that the factors of I_7 must be of one of the forms

$$280k + 1, 43, 71, 169, 197, 239, 253, 267.$$

There are 168 odd primes less than the square root of I_7 but by using this elimination process one factor (239) is disclosed at the fourth trial. The other factor (4649) is equal to $(280 \times 16) + 169$.

The following table has been constructed on these lines and should prove useful to anyone intending to pursue the subject.

Table 11

'Forms' of Factors of I_n

n								
5	$40k+1$,	31.						
7	$280k+1$,	43,	71,	169,	197,	239,	253,	267.
9	$360k+1$,	37,	163,	199,	253,	271,	289,	307.
11	$440k+1$,	67,	89,	111,	133,	199,	243,	397.
13	$520k+1$,	27,	53,	79,	157,	209,	313,	391.
15	$120k+1$,	31.						
17	$680k+1$,	239,	307,	409,	443,	477,	511,	613.
19	$760k+1$,	39,	77,	191,	267,	533,	609,	723.
21	$840k+1$,	43,	169,	253,	547,	631,	757,	799.
23	$920k+1$,	73,	231,	277,	323,	369,	507,	599.
25	$200k+1$,	151.						
27	$1080k+1$,	163,	271,	649,	757,	919,	973,	1027.
29	$1160k+1$,	117,	523,	639,	813,	871,	929,	987.
31	$1240k+1$,	187,	249,	311,	373,	559,	683,	1117.
33	$1320k+1$,	67,	133,	199,	397,	529,	991,	1123.
35	$280k+1$,	71.						
37	$1480k+1$,	519,	667,	889,	963,	1037,	1111,	1333.
39	$1560k+1$,	79,	157,	391,	547,	1093,	1249,	1483.
41	$1640k+1$,	83,	329,	493,	1067,	1231,	1477,	1559.
43	$1720k+1$,	173,	431,	517,	603,	689,	947,	1119.
45	$360k+1$,	271.						
47	$1880k+1$,	283,	471,	1129,	1317,	1599,	1693,	1787.
49	$1960k+1$,	197,	883,	1079,	1373,	1471,	1569,	1667.

(7) Even with the aids described above, however, the factorisation of I_n and J_n becomes extremely difficult as n increases and there is scope here for further research into links between possible prime factors and n. As a start in this direction it might be worth while to note that although I_n is not always composite it is obvious that every prime is a factor of *some* I_n (see the opening remarks on recurring decimals in Chap. 3). It may therefore be of interest to approach the problem, empirically, from the rear.

In the following table the odd primes up to 1051 are listed, together with the n values of I_n for which they are factors. Thus, for example, it will be seen that when $p = 41$, the corresponding

Table 12

Values of n for which I_n is a multiple of p

p	n	p	n	p	n	p	n	p	n
3	3	163	81	367*	366	593*	592	823*	822
7*	6	167*	166	373	186	599	299	827	413
11*	2	173	43	379*	126	601	300	829*	276
13	6	179*	178	383*	382	607*	202	839	419
17*	16	181*	180	389*	388	613	51	853	213
19*	18	191	95	397	99	617*	88	857*	856
23*	22	193*	192	401	200	619*	618	859*	26
29*	28	197	98	409	204	631	315	863*	862
31	15	199	99	419*	418	641	32	877	438
37	3	211*	30	421*	140	643	107	881	440
41	5	223*	222	431	215	647*	646	883	441
43	21	227	113	433*	432	653	326	887*	886
47*	46	229*	228	439	219	659*	658	907	151
53	13	233*	232	443	221	661*	660	911	455
59*	58	239	7	449	32	673*	224	919	459
61*	60	241	30	457*	152	677	338	929	464
67	33	251*	50	461*	460	683	341	937*	936
71	35	257*	256	463*	154	691*	690	941*	940
73*	8	263*	262	467	233	701*	700	947	473
79	13	269*	268	479	239	709*	708	953*	952
83	41	271	5	487*	486	719	359	967*	322
89	44	277	69	491*	490	727*	726	971*	970
97*	96	281	28	499*	498	733	61	977*	976
101*	4	283	141	503*	502	739*	738	983*	982
103*	34	293	146	509*	508	743*	742	991	495
107	53	307	153	521	52	751	125	997	166
109*	108	311	155	523	261	757	27	1009	252
113*	112	313*	312	541*	540	761	380	1013	253
127*	42	317	79	547	91	769	192	1019*	1018
131*	130	331*	110	557	278	773	193	1021*	1020
137*	8	337*	336	563	281	787	393	1031	103
139*	46	347	173	569	284	797	199	1033*	1032
149*	148	349*	116	571*	570	809	202	1039	519
151	75	353*	32	577*	576	811*	810	1049	524
157	78	359	179	587	293	821*	820	1051*	1050

value of n is 5, indicating that 41 is an exact divisor of I_5, or 11,111.

(It is worth noting that when p is of one of the forms $40k + 7$, 11, 17, 19, 21, 23, 29, 33, then n is always an even number and consequently p is also a factor of $J_{n/2}$.)

(A still more important observation can be divided into two parts:

(*a*) When p is of the above form, $40k + 7$, 11, 17, etc.—marked thus * in the table—then n is equal to $(p - 1)$ divided by an *odd* number. For example, $(17 - 1)/1 = 16$; $(73 - 1)/9 = 8$, etc.

(*b*) When p is of the form $40k + 1$, 3, 9, 13, 27, 31, 37, 39, n is equal to $(p - 1)$ divided by an *even* number, as $(31 - 1)/2 = 15$; $(173 - 1)/4 = 43$.)

Note also: $p =$ 493121,	$n =$	67 (Brillhart)
497867		89 (Brillhart)
18797		127 (McCullough)
80173		131 (McCullough)
2467		137 (McCullough)
12517		149 (McCullough)
12671		181 (McCullough)
52009		197 (McCullough)

6 PATTERNS AMONG THE 'ONES'

(1) We saw in the last chapter that when n is composite it is always a simple matter to find at least two factors of I_n and although it may not be easy to say whether these are prime or composite the process does reduce the magnitude of the numbers to be tested. The examples given employed only the digits 0, 1 and 9, but other factor patterns can be found when n is of certain specific 'forms'.

For instance, we have:

(*A*) When $n = 3k$,

$I_3 = 3.37$
$I_6 = 33.3367$
$I_9 = 333.333667$
$I_{12} = 3333.33336667$ and so on.

(*B*) When $n = 6k$,

$I_6 = 91.1221$
$I_{12} = 9901.11222211$
$I_{18} = 999001.111222222111$
$I_{24} = 99990001.111122222222221111$ etc.

(*C*) When $n = 4k + 2$,

$I_2 = 11$
$I_6 = 91.1221$
$I_{10} = 9091.122221$
$I_{14} = 909091.12222221$
$I_{18} = 90909091.1222222221$ etc.

Associated with the above the following curiously regular 'irregularities' are worth noting:

3367 = 37.91
33336667 = 37.900991
3333366667 = 37.90090991
33333336666667 = 37.900900990991
3333333366666667 = 37.90090090990991
33333333336666666667 = 37.900900900990990991
3333333333336666666666667 = 37.90090090090990990990990991 etc.

A more intriguing partition of factors of certain I_n numbers is to be found in a system which is not completely consistent. When $n = 5k$, I_n can be divided into the following pairs of factors:

n	
5	41.271
10	451.2463661
15	4551.24414658561
20	45551.243926831707561
25	455551.2439048780731709756 + $\underline{55555}$
30	4555551.24390268292707317097561
35	45555551.243902463414636585366097561
40	455555551.2439024414634146585365856097561
45	4555555551.24390243926829268317073170756097561
50	45555555551.243902439048780487807317073170975609756 + $\underline{5555555555}$

and so on, with every I_{25k} failing to conform to the pattern. (Note that $455551 = 41^2 . 271$.)

This peculiarity is not confined to members of the I_{5k} clan as can be seen by making a similar partition when $n = 7$.

n	
7	239.4649
14	2629.4226364059
21	26529.4188288707117159
28	265529.4184518870297071548159
35	2655529.4184142259832640167405858159
42	26555529.4184104602514644355648539748958159
49	265555529.4184100836820125523016736402092050251046 + $\underline{7777777}$

and so on. Again, $265555529 = 239^2 . 4649$.

(2) Similar investigations into the construction of the J numbers must, I am sure, be equally rewarding and I recommend this as a project for the reader. In case however, the large numbers with which we have been dealing have been a little over-powering this chapter will be rounded off with a few observations which do not demand quite so much muscular mathematics.

It may seem hardly worth remarking that since 41, for instance, is a divisor of 11111, it must also divide 4111111, but it is perhaps not quite so obvious that it is also a factor of 4000001, 400000000001, etc. Similarly 37 is an exact divisor of 31117, 31111117, . . . and 30007, 30000007, etc.

The reasons for these 'phenomena' are easily discovered but what follows is not so obvious. Since, for example, 4649 is a factor of I_7 it is also an exact divisor of 40000000649, 46000000049, 46400000009 and all other members of the same pattern containing $7k$ zeros (or for that matter $7k$ 'ones').

(3) We have seen that the I and J numbers can present problems of interest and considerable complexity and that the determination of their prime or composite character is in general extremely difficult; on the other hand their construction by additive methods is, as might be expected, not only simple but open to a variety of approaches. Some of these relating to the I numbers are given below; the reader may find it diverting to extend these examples or perhaps to find similar expressions for the J numbers.

(A) $1 = (6 - 5)/1 = (7 - 4)/3 = (8 - 3)/5 = (9 - 2)/7 \ldots$
$11 = (6^2 - 5^2)/1 = (7^2 - 4^2)/3 = \ldots$
$111 = (56^2 - 55^2)/1 = (57^2 - 54^2)/3 = \ldots$
$1111 = (556^2 - 555^2)/1 = (557^2 - 554^2)/3 = \ldots$ and so on.

(B) $11 = (10^2 - 1)/9$
$111 = (60^2 - 51^2)/9$
$1111 = (560^2 - 551^2)/9$
$11111 = (5560^2 - 5551^2)/9$
$111111 = (55560^2 - 55551^2)/9$ etc.

(*C*)

$$\begin{aligned} 11 &= ((9 \times 9) + 7)/8 \\ 111 &= ((98 \times 9) + 6)/8 \\ 1111 &= ((987 \times 9) + 5)/8 \\ 11111 &= ((9876 \times 9) + 4)/8 \\ \ldots &\quad \ldots \\ 111111111 &= ((98765432 \times 9) + 0)/8 \\ 1111111111 &= ((987654321 \times 9) - 1)/8 \end{aligned}$$

(*D*)

$$\begin{aligned} 11 &= 1 \times 9 + 2 \\ 111 &= 12 \times 9 + 3 \\ 1111 &= 123 \times 9 + 4 \\ 11111 &= 1234 \times 9 + 5 \\ \ldots &\quad \ldots \\ 111111111 &= 12345678 \times 9 + 9 \\ 1111111111 &= 123456789 \times 9 + 10 \end{aligned}$$

(*E*)

$$\begin{aligned} 11 &= 6^2 - 5^2 = 4^2 - 5 = 3^2 + 2 \\ 1111 &= 56^2 - 45^2 = 34^2 - 45 = 33^2 + 22 \\ 111111 &= 556^2 - 445^2 = 334^2 - 445 = 333^2 + 222 \\ 11111111 &= 5556^2 - 4445^2 = 3334^2 - 4445 = 3333^2 + 2222 \text{ etc.} \end{aligned}$$

(*F*)

$$\begin{aligned} 111 &= 6^2 + 75 = 7^2 + 62 \\ 11111 &= 66^2 + 6755 = 67^2 + 6622 \\ 1111111 &= 666^2 + 667555 = 667^2 + 666222 \\ 111111111 &= 6666^2 + 66675555 = 6667^2 + 66662222 \quad \text{etc.} \end{aligned}$$

(*G*)

$$\begin{aligned} 11 &= (3 \times 4) - 1 \\ 1111 &= (33 \times 34) - 11 \\ 111111 &= (333 \times 334) - 111 \\ 11111111 &= (3333 \times 3334) - 1111 \quad \text{etc.} \end{aligned}$$

(*H*)

$$\begin{aligned} 1 &= 1 \\ 11 &= 2 + 9 \\ 111 &= 3 + (3 \times 9) + 9^2 \\ 1111 &= 4 + (6 \times 9) + (4 \times 9^2) + 9^3 \\ 11111 &= 5 + (10 \times 9) + (10 \times 9^2) + (5 \times 9^3) + 9^4 \quad \text{etc.} \end{aligned}$$

Here, in effect, I_n has been expressed in the scale of nine, and it will be seen that the coefficients are those of the binomial expansion of $(1 + x)^n$.

If should be noted—though I hope without surprise—that if the sequences of numbers defined by $2^n \pm 1$ are expressed in the scale of 2, they become:

	$2^n - 1$		$2^n + 1$		
n	S_{10}	S_2	S_{10}	S_2	
1	1	1	3	11	
2	3	11	5	101	
3	7	111	9	1001	
4	15	1111	17	10001	etc.

And finally, although it is outside the intended scope of this book I cannot resist adding this little gem for the entertainment of those readers who are familiar with the operation of determinants:

$$\begin{vmatrix} I_n & I_{n+1} & I_{n+2} \\ I_{n+1} & I_{n+2} & I_{n+3} \\ I_{n+2} & I_{n+3} & I_{n+4} \end{vmatrix} = 0$$

In expanded, and far less picturesque form, this becomes (for compactness putting $o, p, q, r,$ in the place of $n + 1, n + 2,$ etc.):

$$I_n(I_p \times I_r - I_q^2) + I_o(I_q \times I_p - I_r \times I_o) + I_p(I_o \times I_q - I_p^2) = 0$$

Testing this for $n = 1$, for example, we then have:

$$\begin{aligned} &(111.11111 - 1111^2) + 11(1111.111 - 11111.11) + \\ &\quad 111(11.1111 - 111^2) = (1233321 - 1234321) + \\ &\quad 11(123321 - 122221) + 111(12221 - 12321) = \\ &\quad -1000 + 11(1100) + 111(-100) = -1000 + \\ &\quad 12100 - 11100 = 0. \end{aligned}$$

EXERCISES

1. What factors are common to both 1001001001 and 100010001?
2. Why must 90909091 be composite?
3. What are the factors of 33336667?

7 PRIME NUMBERS

(1) Primes, those integers which have no factors other than themselves and unity, have fascinated and tortured amateurs and some of the greatest mathematical minds in history alike for centuries. And the total of our knowledge of them is still disappointingly small.

We know that the primes continue indefinitely—that there is no 'largest' prime; we know that no algebraic expression can be written that generates only primes. We know an analytic formula which will tell us how many primes to expect less than any given number, but only to a very rough approximation, and a procedure giving an exact figure at the expense of extremely laborious computations. We know of some expressions in x (variable) that contain infinitely many primes but these are mostly trivial and though we suspect others they have not yet been proved.

A few tests which distinguish between prime and composite numbers are known but they are difficult to apply to integers of even moderate size. And, broadly speaking, that's about the lot.

Oddly enough, in spite of what has been said, we can compose strings of consecutive integers which are composite for as many terms as we like to specify although to be frank this is of little value in the theory of primes.

Having said this let us look at some of the intriguing ways in which the problem of the primes has been attacked.

Proof of the infinity of the primes dates back at least to Euclid whose reasoning is a model for all time of succinct elegance. Suppose, he argued, that P is the largest prime: then if P and all the smaller primes are multiplied together and 1 is added to the product, then this new number cannot be exactly divisible by P or any lesser prime. It must therefore either be itself a prime or divisible by some prime greater than P; in either case there is a prime greater than P.

Although there can be no formula which produces only primes—

proof of this may be found in any college algebra—there are some expressions which have an almost magical facility up to a point. For instance, the sequence $p + 0, p + 2, p + 6, \ldots$ (which is the same thing as $p + n^2 - n$, n taking the successive values $1, 2, 3, \ldots$) obviously becomes equal to n^2 when $n = p$ but up to this point, when $p = 3, 5, 11, 17$, and 41 we have:

$p = 3$	5	11	17	41		
5	7	13	19	43	347	1163
9	11	17	23	47	383	1231
	17	23	29	53	421	1301
	25	31	37	61	461	1373
		41	47	71	503	1447
		53	59	83	547	$1523 = 39^2 + 2$
		67	73	97	593	$1601 = 40^2 + 1$
		83	89	113	641	$1681 = 41^2$
		101	107	131	691	
		121	127	151	743	
			149	173	797	
			173	197	853	
			199	223	911	
			227	251	971	
			257	281	1033	
			289	313	1097	

All these integers except the terminal ones are prime. It is difficult to believe that it is only by chance that these sequences pick their steps so daintily without once treading on a composite number but so far no one has been able to find a prime greater than 41 which produces a further series of this kind.

Other expressions are occasionally 'discovered', but these arise from the fact that n can be given negative values or be replaced by, for example $(n - 40)$, and in this way it is possible to devise a confusing array of 'different' formulae. Thus, for instance, we can write:

$$n^2 + 3n + 43$$
$$n^2 + 5n + 47$$
$$n^2 + 7n + 53 \quad \text{and so on,}$$

or $$n^2 - 79n + 1601$$

or $$n^2 + an + \frac{(a^2 + 163)}{4}. \quad \text{(when } a \text{ is odd).}$$

All these formulae produce exactly the same sequence of prime numbers as the original.

(2) A fundamental property of prime numbers is conveyed in the statement, 'If n is any integer and p is a prime, then $n^p - n$ is exactly divisible by p.'

Over 2000 years ago the Chinese knew this to be true in the special case of $n = 2$, but nowadays the above is known as 'Fermat's (lesser) Theorem'. Fermat first declared the general form—without proof—in a letter to a fellow mathematician in October 1640. Removing the factor n we obtain the form in which the theorem is more commonly stated, namely; 'If p is a prime number, and n an integer not a multiple of p, then $n^{p-1} - 1$ is exactly divisible by p.'

The first formal proof of this theorem was given by Leibniz (1646–1716). In an age in which algebraic symbolism was in a primitive state and which consequently required propositions of this sort to be argued out verbally the proof must have presented formidable difficulty: today it is disposed of in a few lines in any school algebra.

The theorem permiates the Theory of Numbers like a mycelium but unfortunately its converse cannot be relied upon as a test of primality without some highly sophisticated refinements.

Another important theorem connected with prime numbers is that known as 'Wilson's'. This, reminiscent of Euclid's proof of the infinity of primes, states that if, *and only if*, p is a prime, then

$$(2.3.4.5.\ldots.(p-1)) + 1$$

is exactly divisible by p. ($2.3.4.5.\ldots.n$ is known as 'factorial n' and is written $n!$). In most Algebra schoolbooks the theorem is stated thus:

If p is a prime number, then $(p - 1)! + 1$ is divisible by p. Contrary to Fermat's this theorem is only true when p is prime as can easily be shown. For if p has a factor, say m, then m is less than p and must therefore divide $(p - 1)!$. The same goes for

its other factor and hence neither of them can be divisors of $(p - 1)! + 1$.

Factorial numbers increase very rapidly even for quite small values of n, as can be seen from the table in the Appendix, and consequently Wilson's theorem has no value at present as a test of primality. On the other hand it has some interesting consequences. For instance it is not difficult to derive from it that if p and $p + 2$ are both prime then

$$2(p-1)! + 1 \text{ is divisible by } p + 2.$$

Anything bearing on 'pairs' of primes such as 5/7, 11/13, 17/19, 59/61, 115301/115303, 100004561/100004563, 1000000009649/1000000009651, etc., is worth noting since the theory of these numbers has scarcely been scratched and it is not even known whether they continue indefinitely.

Perhaps if some technique for dealing with large factorials can be developed, comparable say with that now available for use against the converse of Fermat's theorem, Wilson's might yet disclose new horizons.

Although this property of factorials does not appear to provide an effective weapon against the primes factorials do enable us to make the most fantastic demands upon composite numbers. Lists of prime numbers display irregularly spaced gaps indicating strings of consecutive composite integers; these appear to be randomly distributed and increase in length as the numbers become larger. Nevertheless it would require a long list of primes and quite a laborious search before we could find say, fifty numbers without a single prime among them. And yet if we were to make the impossible-looking demand for a million consecutive composite numbers factorials provide the answer immediately. It is only necessary to write $1000001! \pm 2, \pm 3, \ldots \pm 1000001$, and we have two sets of numbers which fulfil the requirement. For since $n!$ is divisible by 2, 3, 4, . . ., adding or subtracting 2, 3, 4, etc., will not affect this property and all integers within the intervals proscribed must therefore be composite.

Of course this is a most profligate means of ensuring that such a demand is met. For instance to make sure of say, twenty-four consecutive composite numbers it is necessary to start with 25! or 15511210043330985984000000 whereas all the integers between 3137 and 3163 are composite.

(3) A most interesting and far-reaching generalisation of Fermat's Theorem was first announced by Euler in 1760. In this he began by examining those integers which are 'relatively prime' to one another, that is which have no common factors. Thus, for example, 7, 8, 9, 25, are all relatively prime though three of them are composite.

Now any given number m has just so many integers less than m which are relatively prime to it; the number of such integers is denoted by $\phi(m)$—Euler's function. When m is a prime then clearly $\phi(m) = m - 1$, but with m composite there is some counting to be done. Thus when $m = 14$, three are six integers prime to m, namely, 1, 3, 5, 9, 11, 13, and therefore $\phi(14) = 6$.

Euler's generalisation in effect substitutes $\phi(m)$ for $(p - 1)$ in Fermat's statement that $n^{p-1} - 1$ is divisible by p, and extends the theorem to include *all* values of m. We now have, "$n^{\phi(m)} - 1$ is exactly divisible by m, provided only that n is relatively prime to m".

Taking again the above example, $m = 14; \phi(m) = 6$, the theorem shows that for $n = 1, 3, 5, 9, 11, 13; n^6 - 1 = 14k$.

Thus: $3^6 - 1 = 728 = 14.52$
$5^6 - 1 = 15624 = 14.1116$
$9^6 - 1 = 531440 = 14.37960$ etc.

It should be noted that n can be *any* integer that is relatively prime to m, and thus we have for instance,

$$17^6 - 1 = 24137568 = 14.1724112.$$

(Note. Since 7 is not a member of the set, $7^6 - 1 = 117648 \neq 14k$.)

Although quite incidental to the above ideas it will be of interest to observe that if any two or more of the integers 1, 3, 5, 9, 11, 13, are multiplied together and the product divided by 14, the remainder is always a member of the set.

For example, $5.13 = 65 = (4.14) + 9$,
$3.11 = 33 = (2.14) + 5$, etc.

This is a property of all sets of integers which are relatively prime to a given number.

Methods of determining $\phi(m)$ for any value of m can be stated simply. Their derivations and proofs will not, however, be given here because they are most elegantly presented by the use of congruence techniques. (See H. Davenport's *The Higher Arithmetic*.) The basic principle can be described as follows:

Let $m = a^r . b^s . c^t \ldots$, where a, b, and c, etc., are prime factors.

Then $\phi(m) = m(1 - 1/a)(1 - 1/b)(1 - 1/c) \ldots$. (Note that r, s, t, etc. do not enter into the calculation.) Thus, for example, when $m = 40 = 2^3 \times 5, \phi(40) = 40(1 - 1/2)(1 - 1/5) = 40 \times 1/2 \times 4/5 = 16$.

In the following list (Table 13) all the values of $\phi(m)$ for $m < 400$ have been tabulated. Before looking at its practical use as an aid to factorisation the following observations can be made; all but the final one can be proved and can be used to extend the table as required.

(*a*) Every value of $\phi(m)$ is an even number.
(*b*) When m is prime, then $\phi(m) = m - 1$.
(*c*) When m is odd, then $\phi(m) = \phi(2m)$.
(*d*) When m is even, then $\phi(2m) = 2.\phi(m)$.
(*e*) When m is odd yet not a multiple of three, (i.e. of the form $6k \pm 1$), then $\phi(3m) = \phi(4m) = \phi(6m)$.
(*f*) When m is a square number, say a^2, then $\phi(a^2) = a.\phi(a)$.
(*g*) All values of $\phi(m)$ are repeated at least once. This is a conjecture which has not yet been proved.

The practical use to which this function can be put is perhaps best illustrated by taking an example from Table 13. It will be seen, for instance, that $\phi(399) = 216$. (See the end of the last line in the table.) Since 10 is prime to 399, we can therefore state that $10^{216} - 1$, $(= 9.I_{216})$ is a multiple of 399, $(= 3.7.19)$. Looking further up the table it will be seen that 216 also appears as a function of 351, 333, 327, 259, and 247. Factorising these numbers we obtain the 'new' primes 13, 37, 109 and it is then possible to state that I_{216} is a multiple of $3.7.13.19.37.109$. (The reader might find a gentle exercise in checking these findings back to the original definition of Euler's function.) It will be noted that Fermat could not have helped here, since 217 is not prime.

(4) It was recognised early in mathematical history that no formula could be found that excluded all composite numbers but there still remained conjectures about 'forms' of numbers which (*a*) provided

Table 13

Euler's Function $\phi(m)$

($\phi(m)$ is the number of integers less than, and relatively prime to m)

m	0	1	2	3	4	5	6	7	8	9
0	—	—	1	2	2	4	2	6	4	6
10	4	10	4	12	6	8	8	16	6	18
20	8	12	10	22	8	20	12	18	12	28
30	8	30	16	20	16	24	12	36	18	24
40	16	40	12	42	20	24	22	46	16	42
50	20	32	24	52	18	40	24	36	28	58
60	16	60	30	36	32	48	20	66	32	44
70	24	70	24	72	35	40	36	60	24	78
80	32	45	40	82	24	64	42	56	40	88
90	24	72	44	60	46	72	32	96	42	60
100	40	100	32	102	48	48	52	106	32	108
110	40	72	48	112	36	88	56	72	58	96
120	32	110	60	80	60	100	36	126	64	84
130	48	130	40	108	66	72	64	136	44	138
140	48	92	70	120	48	112	72	84	72	148
150	40	150	72	96	60	120	48	156	78	104
160	64	132	54	162	80	80	82	166	48	156
170	64	108	84	172	56	120	80	116	88	178
180	48	180	72	120	88	144	60	160	92	108
190	72	190	64	192	96	96	84	196	60	198
200	80	132	100	168	64	160	102	132	96	180
210	48	210	014	140	106	168	64	180	108	444
220	220	80	192	72	222	96	120	112	226	72
230	88	120	112	232	72	184	116	156	96	238
240	64	240	110	162	120	168	80	216	120	164
250	100	250	72	220	126	128	128	256	84	216
260	96	168	130	262	80	208	108	176	132	268
270	72	270	128	144	136	200	88	276	138	180
280	96	280	92	282	140	144	120	240	96	272
290	112	192	144	292	84	232	144	180	148	264
300	80	252	150	200	144	240	96	306	120	204
310	120	310	96	312	156	144	156	316	104	280
320	128	212	132	288	108	240	162	216	160	276
330	80	330	164	216	166	264	96	336	156	224
340	128	300	108	294	168	176	172	346	112	348
350	120	216	160	352	116	280	176	192	178	358
360	96	342	180	220	144	288	120	366	176	240
370	144	312	120	273	160	184	184	336	108	378
380	144	252	190	382	128	240	192	252	192	388
390	96	352	168	260	196	312	120	396	198	216

only primes, and (*b*) contained an infinity of primes. Perhaps the best known guess in class (*a*) was that made by Fermat. His formula for primes, to this day known as 'Fermat Numbers' or F_n was $2^{2^n} + 1$, where $n = 1, 2, 3, \ldots$. This expression clearly produced primes for $n = 1, 2, 3, 4$, and there the matter stood until in 1732 Euler showed that $F_5 = 4{,}294{,}967{,}297 = 641 . 6{,}700{,}417$.

Nearly 150 years elapsed before F_6 was cracked.

$$F_6 = 18{,}446{,}744{,}073{,}709{,}551{,}617 = 274{,}177 \times 67{,}280{,}421{,}310{,}721.$$

Many more composite numbers of this form have since been found but not, so far as I know, any more primes. Thus F_n for $n = 7, 8, 10, 13, 14, 16$, is known to be composite, and the following factors have been found:

F_9 is divisible by $37.2^{16} + 1$
F_{11} is divisible by $(39.2^{13} + 1)(119.2^{13} + 1)$
F_{12} is divisible by $(7.2^{14} + 1)(397.2^{16} + 1)$
F_{15} is divisible by $579.2^{21} + 1$
F_{18} is divisible by $13.2^{20} + 1$
F_{23} is divisible by $5.2^{25} + 1$
F_{36} is divisible by $5.2^{39} + 1$
F_{38} is divisible by $3.2^{41} + 1$
F_{73} is divisible by $5.2^{75} + 1$

(To appreciate the size of these numbers the last *factor*, $5.2^{75} + 1$ is equal to 188,894,659,314,785,808,547,841.)

These numbers increase at a fabulous rate and yet, thanks to highly sophisticated techniques developed in the last half century, it is now known that the almost incomprehensible number F_{1945} is divisible by $5.2^{1947} + 1$.

It would seem that Fermat's conjecture has somersaulted and one should now ask whether any primes beyond F_4 exist at all.

(5) Although not strictly belonging to either of the above classes, no discussion of the primes would be complete without some mention of the 'Mersenne' numbers. Mersenne, a contemporary of Fermat (1588–1648), studied the numbers $2^p - 1$ (p—prime) and in 1644 made a statement to the effect that the only values of p that make $2^p - 1$ a prime are 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257. (Being nobody's fool one assumes that he added, or implied, $p < 260$.)

These numbers do not increase with anything like the rapidity of Fermat's, but nevertheless the factorisation of the intervening numbers must have presented, in those days, a most formidable problem. It was not until the end of the nineteenth century that M_{61} was proved to be prime and this was followed by Cole's demonstration in 1903 that M_{67} was composite (193,707,721 × 761,838,257,287). More primes of this form, $p > 257$, have been found since, the largest to date (1968) being $2^{11213} - 1$.

Mathematical fortune-telling on meagre empirical evidence is no longer a popular pastime but it must not be forgotten that the efforts to prove or disprove these two wild guesses have probably contributed more to the development of factorisation methods and tests of primality than any other problems in the Theory of Numbers.

In class (*b*) we have a rather different situation. Since, for instance, $4n \pm 1$ and $6n \pm 1$ contain respectively all the odd numbers and all those not divisible by three, these progressions obviously contain all the primes. Simple as it may seem the proof that each of these four progressions contains an infinity of primes requires some effort and any amateur who completes the set can call himself a mathematician.

Dirichlet proved (1837) that the series $an + b$ (a relatively prime to b) generates an infinite number of primes and it is conjectured—as yet without proof—that the same applies to the series $n^2 + 1$. There is, of course, no limit to propositions of this kind but when it comes to proof it is difficult to find a point of contact with the few known properties of the primes.

EXERCISES

1. For what values of n is $n^8 - 1$ divisible by 20?
2. Use Table 13 to find four prime factors of $5^{12} - 1$.
3. What is the value of $\phi(402)$?

8 PRIME NUMBERS (Part 2)

(1) In this chapter some of the lesser known surmises and observations concerning the primes will be described.

One of the most curious of these is Euler's proposed test for the primality of integers of the form $N = 4n + 1$ which end in 3 or 7, or more concisely those of the form $20m + 13, + 17$. (These integers represent one quarter of all the odd numbers.) The criterion then runs:

'Let R be the remainder after subtracting from $2N$ the next smaller square which ends in 5, namely $(5n)^2$. Then to R add cumulatively the numbers $100(n - 1), 100(n - 3), 100(n - 5), \ldots$ and so on. If among these sums, including R itself, there occurs a *single* square number, then N is either a prime or is divisible by this square. If, on the other hand, two or more squares, or none, occur then N is composite.'

The following four examples will serve to disentangle these unusual and somewhat confusing instructions.

$$(A)\ N = 637. \quad \begin{aligned} 2N &= 1274 \\ 35^2 &= \underline{1225} = (5n)^2, \text{ so } n = 7. \\ R &= \ \ \ 49 \end{aligned}$$

Then R, $(R + 600)$, $(R + 400)$, $(R + 200) = 49, 649, 1049, 1249$. In this set the only square is 49, and therefore N is either prime or is divisible by 49. In fact $637 = 49 \cdot 13$.

$$(B)\ N = 2437. \quad \begin{aligned} 2N &= 4874 \\ 65^2 &= \underline{4225} \text{ therefore } n = 13 \\ R &= \ \ 649 \end{aligned}$$

The sequence then becomes, 649, 1849, 2849, 3649, 4249, 4649, 4849. Here the only square number is 1849 and since 2437 is clearly not divisible by 1849 then it is prime.

$$(C)\ N = 2303. \quad \begin{aligned} 2N &= 4606 \\ 65^2 &= \underline{4225} \quad n = 13 \\ R &= \ \ 381 \end{aligned}$$

Sequence—381, 1581, 2581, 3381, 3981, 4381, 4581. Since this does not contain a square, N is composite.

$$(D)\ N = 2117. \quad \begin{aligned} 2N &= 4234 \\ 65^2 &= \underline{4225} \quad n = 13 \\ R &= \ \ \ \ \ 9 \end{aligned}$$

Sequence—9, 1209, 2209, 3009, 3609, 4009, 4209. Of these numbers 9 and 2209 are square and therefore 2117 is composite ($= 29 \cdot 73$).

One might go on for a long time testing numbers of these two forms at random before coming across an anomaly. In fact where the test indicates that an integer is composite it is invariably true but unfortunately some composite numbers appear amongst those it picks as prime. The first few of these are 153, 333, 477, 657, 833, . . ., and are in general those integers which are multiples of primes of one of the forms $(20k + 13)$ and $(20k + 17)$ and also of one or more of the squares of 3, 7, 9, 11, 23, 27, . . . (i.e. $(20m + 3, 7, 9, 11)^2$).

(Assuming that the first step in any test for primes would be the elimination of 3, 7, 11, 13, as possible factors the first error would not occur until we reached $8993 = 17 \,.\, 23^2$, a truly remarkable achievement.)

Euler's test for primes of the above forms can therefore be polished by the further provision that any integer appearing as a prime must again be tested for divisibility by the numbers $(20m + 3, 7, 9, 11)^2$. In passing it might be of interest to note that the numbers $13^2 \,.\, 17$, $13^2 \,.\, 37$, $13^2 \,.\, 57, \ldots$ have sequences which include *three* squares, whilst $3^2 \,.\, 13$, $7^2 \,.\, 13$, $9^2 \,.\, 13$, $11^2 \,.\, 13, \ldots$ provide the cases where there is *one* square which divides N.

As far as I know no one has put forward anything comparable as a test for the other three forms of odd numbers and this most peculiar approach to the theory of primes—with such a master as its sponsor—might well initiate a rewarding project for some amateur.

(2) The following miscellaneous criteria for primes are given without references; they have been gathered much as one picks blackberries and I cannot say with certainty that all of them have been proved.

They should, however, give some idea of the concentrated effort and the diversity of approach which has been directed upon these fascinating and elusive members of our number system. Perhaps some of them will provoke a renewed attack. In what follows the symbol (*if*) will be used to indicate the expression 'if, and only if'.

(*a*) A number is prime (*if*) it is *not* expressible in the form $ab + xy$ when $a, b, x, y,$ are positive integers such that $a + b = x - y$.

(*b*) A number p is prime (*if*) it occurs $(p - 1)$ times in the $(p - 1)$th set where the first set is 1, 2, 1; the second set is 1, 3, 2, 3, 1—formed by inserting between each two terms of the preceding set their sum—the third 1, 4, 3, 5, 2, 5, 3, 4, 1, and so on.

(*c*) If n is an odd number then $4n + 1$ is prime (*if*) it is a factor of $(2^{2n} + 1)/5$.

(*d*) If p is a prime of the form $4k + 1$ then $2p + 1$ is also a prime (*if*) it exactly divides $(2^p + 1)/3$.

(*e*) If p is a prime of the form $4k - 1$ then $2p + 1$ is also a prime (*if*) it exactly divides $2^p - 1$.

(*f*) If p is an odd prime then the integral part of $(\sqrt{5} + 2)^p - 2^{p+1}$ is exactly divisible by $20p$.

(*g*) A number of the form $6n + 1$ is prime (*if*) n cannot be expressed in either of the forms $6xy + x + y$, or $6xy - x - y$.

(*h*) A number of the form $6n - 1$ is prime (*if*) n is not of the form $6xy + x - y$.

(*i*) If p and q are different primes then $p^{q-1} + q^{p-1} - 1$ is exactly divisible by pq.

(*j*) At least four primes lie between the squares of two consecutive primes (>3).

(*k*) If $4n - 1$ and $8n - 1$ are both primes, then $2^{4n-1} - 1$ is a multiple of $8n - 1$.

(*l*) (*If*) p is prime then $1 + 1/2 + 1/3 + \ldots 1/(p - 1)$ is exactly divisible by p.

(*m*) Given that s and c are respectively the arithmetic means of the squares and cubes of all the numbers less than and relatively prime to n, then $n^3 = 6ns - 4c$.

(*n*) Between any two primes there is an odd number of composite integers.

(3) Although nobody nowadays searches for a formula which will generate nothing but prime numbers it is still reasonable to ask 'what is the number of primes (say $P(x)$) less than a given integer x?'. Most surprisingly one answer was given in 1896 by Hadamard and de la Vallée Poussin independently, both men in or about their thirtieth year. It was to the effect that $P(x)$ approaches the value $x/\log_e x$ as x tends to infinity.

This intrusion of Analysis—which deals with continuous functions and might be called the science of approximation—into a branch of mathematics concerned exclusively with the whole numbers was almost as revolutionary as the introduction of the minus sign into primitive arithmetic.

Of course for some limited values of x, $P(x)$ can be determined by an actual count, and the following lists may be worth noting for reference.

	Number of primes
0 – 1000	168
1000 – 2000	135
2000 – 3000	127
3000 – 4000	120
4000 – 5000	119
5000 – 6000	114
6000 – 7000	117
7000 – 8000	107
8000 – 9000	110
9000 – 10000	112

On a more extended scale we have (excluding 1, and 2) the number of primes less than

10 =	3
100 =	24
1000 =	168
10000 =	1229
100000 =	9592
1000000 =	78498
10000000 =	664579
100000000 =	5761455
1000000000 =	50847478

(4) Finally a word on large primes. About sixteen years ago $2^{127} - 1$, a number of some forty digits, held the record. Shortly afterwards $180(2^{127} - 1)^2 + 1$ (about 83 digits) was proved prime. Since then we have had $2^p - 1$ where $p = 521, 607, 1279, 2203,$ and 2281, this last containing 686 digits.

For a time it was thought by some that $2^q - 1$ was prime when q is a Mersenne prime but it has recently been shown that $2^{2^{13}-1} - 1$ ($= 2^{8191} - 1$) is composite. And now there is this fabulous prime $2^{11213} - 1$, employing 3376 digits.

Even this may have been surpassed by the time these notes appear.

9 AN INTRODUCTION TO DIOPHANTINE EQUATIONS

(1) Equations involving two or more unknowns which require solutions in integers or rational fractions date back to Diophantus of Alexandria in about the third or fourth century A.D. It has taken almost all the intervening time—some fifteen centuries—to bring these problems under control of some sort and even now there still remains the question of whether $x^n + y^n = z^n$, for n greater than 2 has any solution.

Linear Diophantine equations can be expressed in the form $ax \pm by = c$, in which $a, b,$ and c are given and it is required to find integral values of x and y.

We are all familiar with the type of popular mathematical puzzle exemplified by the following: A farmer pays £7 each for sheep and £12 each for calves. (I wouldn't know whether these are realistic prices today.) If he buys some of each for a total bill of £71, how many sheep and calves has he acquired?

Taking the number of sheep and calves to be respectively x and y, we then have the equation $7x + 12y = 71$. It is to be understood of course that at the end of the transaction the animals are to be alive and running about on all four legs.

The standard school-book treatment of such a problem proceeds as follows:

$$7x + 12y = 71 \qquad (1)$$

Dividing by the smaller coefficient (7) we have,

$$x + y + 5y/7 = 10 + 1/7$$

or

$$x + y + (5y - 1)/7 = 10 \qquad (2)$$

Since x and y are integers then $(5y - 1)/7$ must be an integer and consequently $(15y - 3)/7$ is also an integer. (Here a multiplier

is chosen which will make the coefficient of y differ from a multiple of 7 by unity.)

$$\text{Now } (15y - 3)/7 = 2y + y/7 - 3/7$$
$$= 2y + (y - 3)/7$$

As before it follows that $(y - 3)/7$ is an integer and equal, say to n. Then $y - 3 = 7n$ or $y = 7n + 3$ (3)
Substituting this in (2) we have:

$$x + 7n + 3 + (35n + 15 - 1)/7 = x + 12n + 5 = 10$$

or $$x = 5 - 12n.$$

For x to have a positive value n must be zero and we have $x = 5$, and from (2) $y = 3$.

(2) Now there is a much simpler method of arriving at the above result. The algorithm may take a little space to describe but, once understood it is much quicker and more reliable in dealing with equations having large coefficients of x and y.

Taking again the above example we re-write the equation thus: $7x = 71 - 12y$. (This is not really necessary but it helps with the explanation.) Divide the coefficients on the right by 7 and note down the remainders, i.e. $1(c)$ and $-5(b)$. (Remember the general equation is $ax + by = c$.) Set down the numbers 1, 2, 3, . . . $(a - 1)$, or in this case

1, 2, 3, 4, 5, 6

Multiply each of these by the 'b' coefficient, namely 5, to give 5, 10, 15, 20, 25, 30. Divide each of these by 7 and list the remainders thus:

5, 3, 1, 6, 4, 2

Note that the 'c' coefficient, namely 1, is at the third position and write $y = 3$. (As easy as that) $x = 5$ is easily found by substitution.

The process is far simpler to apply than to describe and as a further example of the power of this method we ask 'What are the smallest integers which will satisfy the equation

$$11x + 127y = 1{,}067{,}723\text{?'}$$

Rewriting we have $11x = 1067723 - 127y$. Dividing by 11 the remainders are 8, 6. Then

1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

And, multiplying by 6 and subtracting multiples of 11—

6, 1, 7, 2, 8, 3, . . .

Since 8 is at the fifth position then $y = 5$, and by substitution $x = 97008$.

$$\begin{aligned} \text{Check: } 11 \times 97008 &= 1067088 \\ 5 \times 127 &= \underline{635} \\ & 1067723 \end{aligned}$$

(3) For equations of higher degree the problem becomes more difficult and although some types are amenable to treatment there is in fact as yet no general technique for determining even whether a solution is possible or not.

In the case of equations describing the so called Pythagorean right angled triangles we are on familiar and well trodden ground. These are triangles of sides x, y, and (hypoteneuse) z, and the problem is to find solutions of the equation $x^2 + y^2 = z^2$, x, y, z, being integers and having no factors in common.

The general solution can be reached by the following reasoning—I give this in full because it indicates the best known approach to the more difficult problems in this terrain:

(1) x and y cannot both be even, for then z would also be even.
(2) x and y cannot both be odd for then $x^2 + y^2$ would be of the form $4n + 2$ which is never a square.

Suppose then that x is odd and y, even. We can now write the equation $x^2 + 4t^2 = z^2$ or, $4t^2 = (z + x)(z - x)$.

But x and z are odd and therefore $z + x$, and $z - x$ are both even. Putting $z + x = 2s$ and $z - x = 2r$ we then have $4t^2 = 4sr$, or $t^2 = rs$.

Also $(z + x) + (z - x) = 2s + 2r$ or $z = r + s$ and similarly $x = s - r$.

Since x and z are relatively prime then so are r and s. In consequence the equation $t^2 = rs$ requires that r and s are *each* perfect squares, say $r = n^2$ and $s = m^2$. Then $t = mn$ and it follows that

$$x = m^2 - n^2;\; y = 2mn;\; z = m^2 + n^2.$$

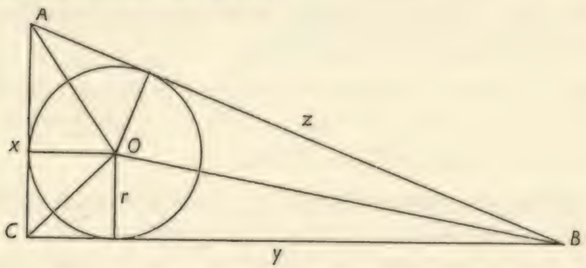
Even so m and n must obey certain conditions. Thus m is greater than n so that x is positive; m and n must have no common factor or it would be shared by x, y, z; m and n must not both be odd for then x and z would both be even.

With these reservations in mind it is possible to tabulate as many solutions—in integers—of the equation $x^2 + y^2 = z^2$ as we wish.

For instance the first few are:

m	n	x	y	z
2	1	3	4	5
3	2	5	12	13
4	3	7	24	25
4	1	15	8	17
5	4	9	40	41
5	2	21	20	29
6	5	11	60	61
6	1	35	12	37

The above general solution to the Pythagorean equation has been well documented but there is a geometrical consequence to this which is perhaps not so well known. Given a right angled triangle whose sides and hypoteneuse can be expressed in integers we consider the radius of the inscribed circle.



The area of $\Delta ABC = x/2y = \Delta s\ AOB + BOC + COA$. Therefore $x/2y = \frac{1}{2}rz + \frac{1}{2}ry + \frac{1}{2}rx = \frac{1}{2}r(x + y + z)$ and $r = xy/(x + y + z)$.

If now we replace x, y, z by their equivalents in m and n we have

$$r = \frac{(m^2 - n^2) \,.\, 2mn}{m^2 - n^2 + 2mn + m^2 + n^2}$$

$$= n(m - n)$$

Since both m and n are integers then r, the radius of the inscribed circle is also always an integer. In particular when $x = 3$, $y = 4$, $z = 5$, then $r = 1$.

EXERCISES

1. Solve $5x + 9y = 1001$, for x and y.
2. If in the equation $x^2 + y^2 = z^2$ we give z the value of 26, what are the corresponding values of x and y?

10 SOME OBSERVATIONS ON X^n

(1) In a broad sense experiments are of two kinds. Those which are made with the object of confirming or disproving a theoretical idea, and those whose sole purpose is that of gathering information. Of the two the latter can usually be expected to provide more opportunities of excitement than the former (unless of course the theory happens to be particularly novel and one's own).

Experiments designed to tell us how numbers behave under certain specific conditions nearly always produce something of interest if they are carried far enough. And in the world of numbers it is indeed remarkable how little initial data is needed to start a train of successive observations.

Let us take as a starting point, for example, one of the simplest properties of integers in the scale of ten. 'All powers of 10 end with the "units" digit 0.' (Incidentally all powers of x in the scale of x do the same but we shall confine ourselves to the everyday system here.) I am sure that no reader will require a formal proof of this. Nor will he need telling that all powers of 5, or for that matter of any number of the form $10k + 5$, end in 5 and that those of 6 end with a 6. The same thing holds for numbers terminating in 1 and it is easily seen that the even powers of $10k + 9$ end in 1 and the odd powers in 9.

Having made these observations it is natural to ask next what happens to the 'units' digits of powers of the remaining numbers $10k + 2, 3, 4, 7,$ and 8. Since we are only concerned at the moment with the final digits it is a simple matter to construct a table without calculating out the powers in full. For instance the units digits of the powers of 2 are 2, 4, 8, 6, 2, 4, etc., those of 3 are 3, 9, 7, 1, 3, and so on. In fact it is quickly seen that the terminal digits of the powers of all the natural numbers repeat themselves after, at most, a cycle of four digits.

Table 14a

Terminal digits of x^n

x	$n = 1$	2	3	4	5
0	0	0	0	0	0
1	1	1	1	1	1
2	2	4	8	6	2
3	3	9	7	1	3
4	4	6	4	6	4
5	5	5	5	5	5
6	6	6	6	6	6
7	7	9	3	1	7
8	8	4	2	6	8
9	9	1	9	1	9

Having produced this table, what can be learnt from it? For a start there are four immediate observations which can be made.

(*A*) All the odd powers of any number x may end in any of the digits 0, 1, 2, 3, . . . 9.

(*B*) x^2, x^6, and in general x^{4n+2} can only end with the digits 0, 1, 4, 5, 6, and 9.

(*C*) x^{4n} is restricted to endings of 0, 1, 5, 6.

(*D*) The even powers of any number can never terminate with the digits 2, 3, 7, or 8.

Much more information, however, can be squeezed out of this table. From at least the time of Fermat and Mersenne, numbers of the general form $x^n \pm 1$ have been the object of much study and speculation particularly in their connection with prime numbers. Although the table has no direct bearing on the primes it does enable us to sweep a wide area free from one class of composite numbers.

Since all integers ending in 0 or 5 are divisible by 5 then clearly the addition of unity to those powers of x which have endings

of 4 or 9 will make the resulting numbers composite (i.e. multiples of 5). Similarly, subtracting one from powers ending in 1 or 6 will have the same result.

Thus $2^2 + 1, 2^6 + 1, \ldots 2^{4+2} + 1$,

and $2^4 - 1, 2^8 - 1, \ldots 2^{4k} - 1$, are all divisible by 5.

Multiples of five are of course easily discernible when expressed in integers but this simple table enables us to list all such multiples which are of the general form $x^n \pm 1$ in the following specific algebraic statements:

$$
\begin{array}{ll}
(10m + 2)^{4k} - 1, & (10m + 2)^{4k+2} + 1, \\
(10m + 3)^{4k} - 1, & (10m + 3)^{4k+2} + 1, \\
(10m + 4)^{2k} - 1, & (10m + 4)^{2k+1} + 1, \\
(10m + 6)^{k} - 1, & (10m + 7)^{4k+2} + 1, \\
(10m + 7)^{4k} - 1, & (10m + 8)^{4k+2} + 1, \\
(10m + 8)^{4k} - 1, & (10m + 9)^{4k+1} + 1. \\
(10m + 9)^{4k} - 1. &
\end{array}
$$

(2) Now in considering the terminal (i.e. units) digits of x^n we have, in effect, been concerned only with the remainders left after division by ten. It will be reasonable to continue with a similar examination of the 'digital roots' of the powers of the natural numbers; these are of course the remainders left after division by nine.

The corresponding table is quite simple to construct; as before it is not necessary to calculate the actual powers of x since if $x = 9r + a$, then $x^2 = 9s + a^2$, $x^3 = 9t + a^3$, etc. Consequently we need only multiply the digital root of x successively by a.

For example when $x = 7$, the digital roots of $x, x^2, x^3, \ldots$ are found thus: 7, $7 \cdot 7 = 49$ ($D.R. = 4$), $7 \cdot 4 = 28$ ($D.R. = 1$) and so on. In this manner we arrive at Table 14b which again is quite general and repeats itself cyclically when extrapolated either vertically or horizontally.

Since all numbers having the digital roots 3, 6, and 9 are divisible by three, we can use this table to sieve out another set of composite values of $x^n \pm 1$.

We see, for example that the digital roots of $2^1, 2^3, 2^5, 2^7$, etc., are always one of the digits 2, 5, 8, and that those of $2^2, 2^4, 2^6, 2^8$, etc., are either 1, 4, or 7. It follows therefore that 3 is always a factor of both

$$2^{2k+1} + 1 \text{ and } 2^{2k} - 1.$$

Table 14b

Digital Roots of x^n

x	$n = 1$	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	2	4	8	7	5	1	2
3	3	9	9	9	9	9	9
4	4	7	1	4	7	1	4
5	5	7	8	4	2	1	5
6	6	9	9	9	9	9	9
7	7	4	1	7	4	1	7
8	8	1	8	1	8	1	8
9	9	9	9	9	9	9	9

Proceeding on these lines and generalising as before we are able then to make a list of all the values of $x^n \pm 1$ which are multiples of 3. Thus:

$$
\begin{array}{ll}
(9m + 2)^{2k} - 1, & (9m + 2)^{2k+1} + 1, \\
(9m + 4)^{k} - 1, & (9m + 5)^{2k+1} + 1, \\
(9m + 5)^{2k} - 1, & (9m + 8)^{2k+1} + 1. \\
(9m + 7)^{k} - 1, & \\
(9m + 8)^{2k} - 1. &
\end{array}
$$

(3) These two tables, extremely simple both in construction and appearance, contain a surprising amount of information, some of which is not easy to uncover by ordinary algebraic processes. To take an elementary example let us assume we are searching for primes among numbers of the form $2^n + 1$. Table 14b tells us that $2^{2k+1} + 1$, an expression containing all the *odd* powers of 2, is divisible by 3, whereas it is seen from Table 14*a* that $2^{4k+2} + 1$, or $2^{2,6,10,\text{etc}} + 1$, is always a multiple of 5. Thus we can only expect to discover primes among numbers of this form when $n = 4k$, i.e. $2^{4k} + 1$. This news will not startle Number Theory addicts but it is still a fair stride from our initial observation that 'all powers of ten have the terminal digit 0'.

The enquiry started at the beginning of this chapter can, of course be carried much further and indeed the examination of the remainders left after dividing x^n by the successive primes 7, 11, 13, . . ., leads into an extremely interesting and advanced field.

However our two primitive tables (14*a* and 14*b*) can still provide a mass of information about exponential numbers which are multiples of 5 and 3.

It will be seen at once, for example, that $x^n + x^{n+2}$ always has the terminal digit 5 (and hence is a multiple of 5) when $x = 10k + 2, 3, 5, 7, 8$. Similarly $2^n + 2^{n+1}$ is obviously a multiple of 3 and this can be extended to $x^n + x^{n+1}$ when $x = 9k + 2, 5, 8$.

The more general numbers $x^n \pm y^m$ can also be separated into multiples of these two factors and a few examples are given here which the reader may care to disentangle for himself.

Multiples of 5.

$$2^{4r+1} + 3^{4s+1}$$
$$2^{4r+1} + 7^{4s-1}$$
$$2^{4r-1} + 8^{4s-1}$$

and as an example of still wider generalisation,

$$(10k + 7)^{4r+1} - (10j + 3)^{4s-1}.$$

where $k, j, r, s,$ take any of the values $0, 1, 2, 3, \ldots$.

Multiples of 3.

$$2^{6r+1} + 5^{6s+2}$$
$$7^{n+1} - 7^n$$

and $(9k + 2)^{6r-1} + (9j + 5)^{6s+2}$.

The use of these tables is not, of course, confined to the addition or subtraction of the powers of only two integers. Thus, for instance, if we consider the horizontal line of digits against $x = 2$, in Table 14*a*, we have,

$$2 + 4 + 8 + 6 + 2 + 4 = 26,$$

and it follows that

$$(2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6) - 1$$

is divisible by five.

In a similar manner Table 14*b* tells us that

$$1^0 + 2^1 + 3^2 + 4^3 + 5^4 + 6^5 + 7^6$$

is a multiple of nine.

The above principles can be applied in a variety of further instances, not all of them trivial, but the point I am hoping to have made is that sometimes the most unexpected findings can emerge from the development of the simplest of elementary observations.

EXERCISES

1. What is (*a*) the 'unit's' digit, and (*b*) the digital root of 11^{11}?
2. In the progression $3^0, 3^1, 3^2, 3^3, \ldots 3^n$, what are the powers which have the remainder 1 after dividing by 7?
3. Given that the sum $1^3 + 2^3 + 3^3 + 4^3 + x^3$ is a multiple of nine, $x\,(<10)$ can have two values. What are they?

11 FINITE ARITHMETIC

(1) In the last chapter we examined a particular class of numbers—the powers of x—with special regard to their remainders after dividing by both 9 and 10. Indeed we started from two simple truths which were obvious by definition, namely that (1) the units digit of any integer remains the same whatever multiple of 10 is added to it, and (2) the 'digital root' of any integer is constant for all additions of $9k$.

We dealt there with two special cases of what is clearly a general statement. If we are considering only the units digits of the integers it might be said that 0 is synonymous with 10, 20, 30, etc., or among digital roots 0 is identical with 9. Similarly it would be just as convenient to mark 0 on a clock face instead of 12, since the cycle starts again at this point. Thus the clock will register the same time in twelve hours or, with perhaps some astronomical or horological reservations, in a thousand years, or any numbers of revolutions. Or to give a further example, whenever two values of n differ by an even number then $(-1)^n$ remains identically the same.

This notion of the 'sameness' of numbers which differ only among themselves by some specified multiple was first crystallised and given a formal mathematical symbol by Gauss towards the end of the eighteenth century. Incidentally, he was in his teens at the time.

What Gauss's notation expresses in fact is that if two integers, a and b, differ by a multiple of a particular number m then a is said to be congruent to b with respect to the modulus m. In this notation we write $a \equiv b \pmod{m}$ instead of the more familiar algebraic $a - b = xm$. Arithmetical congruence then implies equality except for the addition or subtraction of some multiple of m. It will immediately be clear that in the arithmetic 'modulo m' there are no integers greater than $m - 1$, hence the term 'finite arithmetic'.

Congruences have much in common with ordinary equations

and indeed are manipulated according to exactly the same rules with the one proviso that the same modulus is retained throughout the operations.

Thus, if $a \equiv x \pmod{m}$, then, all to $\pmod{m}$, we have

$$ar = xr$$
$$a + r = x + r$$
$$a^r = x^r \quad \text{etc.}$$

If in addition $\quad b \equiv y$

then $\quad ab \equiv xy$

and $\quad a \pm b \equiv x \pm y$

Although a congruence can always be multiplied throughout by an integer, cancellation of a factor is not permissible unless the factor is relatively prime to the modulus. For instance it is legitimate to reduce the congruence $48 \equiv 18 \pmod{10}$ by a factor of 3 to give $16 \equiv 6 \pmod{10}$, but we cannot divide by the even factor 6 (which is not prime to 10) without arriving at the false result $8 \equiv 3 \pmod{10}$.

(2) The concept of congruence is no mere mathematical toy; it enables us to express many propositions with economy and elegance and particularly to operate with numbers so large that ordinary methods of calculation would be impossible to apply.

As an elementary example of this technique it might be interesting to re-examine the 'extraction' of digital roots. Since any integer N can be written $a_0 + 10a_1 + 10^2a_2 + 10^3a_3 + \ldots + 10^na_n$, and $10 \equiv 1 \pmod{9}$—from which it follows that $10^2 \equiv 1$, $10^3 \equiv 1$, $\ldots 10^n \equiv 1$—it is clear that $N \equiv a_0 + a_1 + a_2 + a_3 + \ldots + a_n \pmod{9}$. Hence N is equal to a multiple of 9 plus the sum of its digits and if the latter is divisible by 9 then so is N.

Similar reasoning is equally effective in proving the common test for divisibility by eleven.

For the benefit of any newcomers who find the technique unfamiliar perhaps the following examples will help to clarify the primary methods of operation.

(A) We have $\quad 9 \equiv 2 \pmod{7}$,

then $\quad 9^2 \equiv 2^2 = 4$

and $\qquad 9^3 \equiv 36 \equiv 1,$ and therefore 9^3 divided by 7 leaves the remainder 1.

(*B*) Since
$$6 \equiv -1 \pmod 7$$
$$6^2 \equiv (-1)^2 = 1$$

and hence
$$6^{2n} \equiv 1 \pmod 7$$

(*C*)
$$8 \equiv 1 \pmod 7$$
$$4 \equiv \tfrac{1}{2}$$
$$2 \equiv \tfrac{1}{4}$$

and since $15 \equiv 1 \pmod 7$, and $36 \equiv 1 \pmod 7$, it follows that

$$5 \equiv 1/3 \pmod 7$$
$$3 \equiv 1/5 \pmod 7$$
$$6 \equiv 1/6 \pmod 7$$

It is clear that this arithmetic, although confined by the modulus, is free to operate with both negative signs and fractions.

As an example of the power of the congruence technique we will now apply it to an actual computation. It is required, let us say, to find the remainder upon dividing 5^{26} by 103. Remembering from a previous chapter that any number can be expressed as the sum of single powers of 2, we first note that $5^{26} = 5^2 \times 5^8 \times 5^{16}$.

With this in mind we now proceed as follows:

$$5^2 \equiv 25 \pmod{103}$$
$$5^4 \equiv 625 \equiv 7$$
$$5^8 \equiv 49$$
$$5^{16} \equiv 2401 \equiv 32$$

It is now required to calculate $5^2 \times 5^8 \times 5^{16} \pmod{103}$ and from the above figures we have,

$$5^2 \times 5^8 \equiv 25 \times 49 = 1225 \equiv 92$$
$$92 \times 5^{16} \equiv 92 \times 32 \equiv 2944 \equiv 60.$$

60 then, is the remainder after dividing 5^{26} by 103. To reassure

those who might doubt so facile a result, here is the actual division done the hard way.

```
103/1490116119384765625
     103
     ---
     460  481    446
     412  412    412
     ---  ---    ---
      481  699    345
      412  618    309
      ---  ---    ---
       691  813    366
       618  721    309
       ---  ---    ---
        736  928    572
        721  927    515
        ---  ---    ---
         151   147  575
         103   103  515
         ---   ---  ---
          481   446  60 = remainder.
                     ==
```

(3) It will be of interest at this point to demonstrate the strength of modulus arithmetic by an example of its application to a principle of far reaching importance in Number Theory. Fermat's theorem states that if a is any integer and p a prime, then $a^p - a$ is divisible by p. Or in congruence notation that $a^p \equiv a \pmod p$. Nearly twenty five centuries ago the Chinese knew this to be true in the case of $a = 2$, and until the early nineteenth century the converse, namely that such a factor p is essentially prime, was accepted. Then in 1819 Sarrus discovered that $2^{341} - 2$ is divisible by 341, a composite number (11×31).

The actual computation of the integer $2^{341} - 2$, containing over a hundred digits, and the subsequent tests for divisibility would present a formidable task, but see what light work our modulo arithmetic makes of it.

We have first $\qquad 2^5 \equiv -1 \pmod{11}$

Then it follows that $\qquad (2^5)^{2n} \equiv 1$

and hence $\qquad (2^5)^{68} = 2^{340} \equiv 1$

Similarly $\qquad 2^5 \equiv 1 \pmod{31}$

and $\qquad 2^{340} \equiv 1$

Thus both 11 and 31 are divisors of $2^{340} - 1$ and hence also of $2^{341} - 2$.

Fermat's theorem is perhaps more usually expressed in the terms, 'if p is a prime and a is not a multiple of p then $a^{p-1} - 1$ is divisible by p'. (More refined statements of the theorem will be referred to later.)

That the converse of this is not always true can be seen from a much simpler example than the above. Putting $a = 4$, and $p = 15$, (a composite number) we have $4^{14} - 1 \equiv 0 \pmod{15}$.

Now $\quad 4 \equiv -1 \pmod 5$ and $\quad 4 \equiv 1 \pmod 3$

So $\quad 4^{14} \equiv 1 \quad\quad 4^{14} \equiv 1$

Hence $4^{14} - 1 \equiv 0 \pmod 5$ and $4^{14} - 1 \equiv 0 \pmod 3$. As a check we have $4^{14} - 1 = 268435455$ which is obviously divisible by both 3 and 5.

There are other variants of this famous theorem which lead into much more advanced theory than can be embarked upon here. As examples I will merely comment that the two following statements are true:

$$3^{120} - 1 \equiv 0 \pmod{11^2}$$
$$2^{1092} - 1 \equiv 0 \pmod{1093^2}.$$

(4) Turning now to another development of congruence theory we consider the squares of the natural numbers, $1^2, 2^2, 3^2, \ldots$. Listing the 'digital roots' of these successive squares we have:

n^2	1	4	9	16	25	36	49	64	81	100	121 . . .
D.R.	1	4	9	7	7	9	4	1	9	1	4 . . .

The sequence is clearly repetitive and indeed it is easily shown that however far it is continued no other digits than 1, 4, 7, 9, appear, and since in fact digital roots represent the remainders left after dividing by 9 the set might just as well be written 0, 1, 4, 7. These digits are known in congruence language as 'quadratic residues' modulo 9.

It is easily shown that a closed cycle of this sort will occur with any divisor. Let the remainders (residues) on dividing the successive squares by n be $r_1, r_2, r_3, \ldots$. Then if x is any number we have $(n + x)^2 = n^2 + 2nx + x^2 \equiv x^2 \pmod n$ and $r_n + x$ will have the same value as r_x with a periodic recurrence of the terms. Using a similar argument $(n - x)^2 \equiv x^2 \pmod n$ and hence $r_n - x = r_x$. Therefore for the modulus n there are not more than $\frac{1}{2}n$, or if n is odd, $\frac{1}{2}(n - 1)$, residues to the square numbers.

Quadratic residues to any modulus can be calculated quickly without actually writing down the sequence of square numbers. Since $(x + 1)^2 = x^2 + 2x + 1$ it follows that $r_x + 1 \equiv r_x + 2x + 1 \pmod n$ and for example the residues modulo 13 can be derived in either of the following ways:

x	x^2	r^n	x		r_n
1	1	1	1		1
2	4	4	2		4
3	9	9	3		9
4	16 − 13	3	4	(9 + 2.3 + 1) − 13 =	3
5	25 − 13	12	5	(3 + 2.4 + 1) − 13 =	12
6	36 − 26	10	6	(12 + 2.5 + 1) − 13 =	10
7	49 − 39	10	7	(10 + 2.6 + 1) − 13 =	10
8	64 − 52	12	8	(10 + 2.7 + 1) − 13 =	12
9	81 − 78	3	9	(12 + 2.8 + 1) − 26 =	3
10	100 − 91	9	10	(3 + 2.9 + 1) − 13 =	9
		etc.			etc.

Integers which are not residues (mod n) are known as non-residue and the two classes are connected in the following manner. For any modulus n the product of two or more residues (r^p, r^q) is also a residue. (For if $r^p \equiv x^2$ and $r^q \equiv y^2 \pmod n$ then $r^p r^q = (xy)^2 \pmod n$.) Also the product of two non-residues is again a residue, but the product of a residue and a non-residue is a non-residue.

In the following table (Table 15) quadratic residues of the odd numbers 3, 5, 7, . . . 49, are listed. In order to show the distinction between a prime and composite modulus the table is divided into two sections. (The residues are given in the order of their appearance in the progressive sequence of square numbers.)

Table 15

Quadratic Residues (to n^2)

n	residues
3	1
5	1, 4
7	1, 2, 4
11	1, 4, 9, 5, 3
13	1, 4, 9, 3, 12, 10
17	1, 4, 9, 16, 8, 2, 15, 13
19	1, 4, 9, 16, 6, 17, 11, 7, 3
23	1, 4, 9, 16, 2, 13, 3, 18, 12, 8, 6
29	1, 4, 9, 16, 25, 7, 20, 6, 23, 13, 5, 28, 24, 22
31	1, 4, 9, 16, 25, 5, 18, 2, 19, 7, 28, 20, 14, 10, 8
37	1, 4, 9, 16, 25, 36, 12, 27, 7, 26, 10, 33, 21, 11, 3, 34, 30, 28
41	1, 4, 9, 16, 25, 36, 8, 23, 40, 18, 39, 21, 5, 32, 20, 10, 2, 37, 33, 31
43	1, 4, 9, 16, 25, 36, 6, 21, 38, 14, 35, 15, 40, 24, 10, 41, 31, 23, 17, 13, 11
47	1, 4, 9, 16, 25, 36, 2, 17, 34, 6, 27, 3, 28, 8, 37, 21, 7, 42, 32, 24, 18, 14, 12
9	1, 4, 0, 7
15	1, 4, 9, 1, 10, 6, 4
21	1, 4, 9, 16, 4, 15, 7, 1, 18, 16
25	1, 4, 9, 16, 0, 11, 24, 14, 6, 0, 21, 19
27	1, 4, 9, 16, 25, 9, 22, 10, 0, 19, 13, 9, 7
33	1, 4, 9, 16, 25, 3, 16, 31, 15, 1, 22, 12, 4, 31, 27, 25
35	1, 4, 9, 16, 25, 1, 14, 29, 11, 30, 16, 4, 29, 21, 15, 11, 9
39	1, 4, 9, 16, 25, 36, 10, 25, 3, 22, 4, 27, 13, 1, 30, 22, 16, 12, 10
45	1, 4, 9, 16, 25, 36, 4, 19, 36, 10, 31, 9, 34, 16, 0, 31, 19, 9, 1, 40, 36, 34
49	1, 4, 9, 16, 25, 36, 0, 15, 32, 2, 23, 46, 22, 0, 29, 11, 44, 30, 18, 8, 0, 43, 39, 37

It will be seen from the above table that when n is a prime there are exactly $\frac{1}{2}(n-1)$ residues, all of which are different integers. When n is composite the complete cycle still contains $\frac{1}{2}(n-1)$ residues but some of these are repeated one or more times. When n is a square or contains a square factor, 0 of course appears and may be repeated several times in the complete cycle of residues. The number of different integers comprising the quadratic residues to a given modulus n, tell us then whether n is prime or composite. At first sight this may not seem a very practical method of establishing the primality of n but in fact it provides a very useful technique for detecting not only whether n is composite, but if so what are its factors.

As a simple example let us take $n = 13 \times 37 = 481$. Since all squares less than 481 are residues of n, we have $Q.R.(n) = 1, 4, 9, \ldots 441$ and the sequence can be continued, since $441 = 21^2$, and $(2 \times 21) + 1 = 43$,

$$\begin{aligned} 441 + 43 - 481 &= \ \ 3 \equiv 22^2 \ (\text{mod } 481)\\ 3 + 45 &= 48 \equiv 23^2 \ \text{etc.}\\ 48 + 47 &= 95\\ 95 + 49 &= 144 \ \text{Stop.} \end{aligned}$$

$144 = 12^2$ which has appeared before, and therefore n is composite. Further, since 144 is a residue of the 25th square we now have $25^2 \equiv 12^2$ (mod 481) or $25^2 - 12^2 \equiv 0$. That is $(25-12)(25+12) = 13 \times 37$, the factors of 481. This will be dealt with more fully in a later chapter on factorisation methods.

(5) The study of quadratic residues, to which the above is but the briefest of introductions, has produced many theorems of great importance in the realm of Number Theory but it would not be practicable to advance beyond this stage in a book of this sort.

There are however some observations to be made on the residues to higher powers (than squares) which are neither difficult to grasp nor without their peculiar interest. Consider the residues to a given modulus of numbers of the form x^k, where k takes the successive values $1, 2, 3, \ldots$.

The powers of x increase fairly rapidly as k increases but as noted before, in finite arithmetic it is not necessary to calculate these out in full to obtain the required residues. Thus the residues (mod 5) to 2^k, that is to $1, 2, 4, 8, 16, 32, 64, \ldots$, are obtained simply by multiplying each fresh residue as it is found by 2, discarding of course all multiples of the modulus from the product. In the present case we thus arrive at:

$\underline{1}, 2 \times 1 = \underline{2}, 2 \times 2 = \underline{4}, 2 \times 4 = 8$ and

$8 - 5 = \underline{3}, 2 \times 3 - 5 = 1$

The residues (underlined) are then 1, 2, 4, 3, and the cycle starts again. In other words the kth power residues of 2 to the modulus 5 are 1, 2, 3, 4. 5 is of course a prime but do not be hasty in assuming from this that the residues modulo p always contain the integers $1, 2, 3, \ldots (p-1)$; those of 2^k (mod 7) for instance are 1, 2, and 4, only.

The following tables (16*a*, *b*, *c*) will serve to illustrate some of the properties of the kth power residues; they list the residues to the prime modulo $> 4 < 24$ of the integers 2, 3, and 10 with an extension in the case of 16*c*.

Table 16a

Residues to 2^k

k	(mod) 5	7	11	13	17	19	23
0	1	1	1	1	1	1	1
1	2	2	2	2	2	2	2
2	4	4	4	4	4	4	4
3	3	1	8	8	8	8	8
4	1	2	5	3	16	16	16
5	2	4	10	6	15	13	9
6		1	9	12	13	7	18
7			7	11	9	14	13
8			3	9	1	9	3
9			6	5		18	6
10			1	10		17	12
11				7		15	1
12				1		11	
13						3	
14						6	
15						12	
16						5	
17						10	
18						1	

Table 16b

Residues to 3^k

k	(mod) 5	7	11	13	17	19	23
0	1	1	1	1	1	1	1
1	3	3	3	3	3	3	3
2	4	2	9	9	9	9	9
3	2	6	5	1	10	8	4
4	1	4	4		13	5	12
5		5	1		5	15	13
6		1			15	7	16
7					11	2	2
8					16	6	6
9					14	18	18
10					8	16	8
11					7	10	1
12					4	11	
13					12	14	
14					2	4	
15					6	12	
16					1	17	
17						13	
18						1	

Table 16c

Residues to 10^k

k	(Mod) 7	11	13	17	19	23	31	37	41	43
0	1	1	1	1	1	1	1	1	1	1
1	3	10	10	10	10	10	10	10	10	10
2	2	1	9	15	5	8	7	26	18	14
3	6		12	14	12	11	8	1	16	11
4	4		3	4	6	18	18		37	24
5	5		4	6	3	19	25		1	25
6	1		1	9	11	6	2			35
7				5	15	14	20			6
8				16	17	2	14			17
9				7	18	20	16			41
10				2	9	16	5			23
11				3	14	22	19			15
12				13	7	13	4			21
13				11	13	15	9			38
14				8	16	12	28			36
15				12	8	5	1			16
16				1	4	4				31
17					2	17				9
18					1	9				4
19						21				40
20						3				13
21						7				1
22						1				

* The column (mod 29) has been excluded in order to keep the table reasonably compact (it runs to the full length of 28 integers).

The following observations are not intended to replace the proper theoretical approach to this subject—this is competently dealt with in a number of excellent textbooks—but their object is to whet the appetite and to show some examples of how different branches of arithmetic interlock, sometimes in the most unexpected ways.

Probably the first thing that will be noticed in these tables is that whilst there obviously cannot be more than $(n - 1)$ residues to any given modulus n, the actual number of residues is in many cases only a fraction of $(n - 1)$. What is more, whenever the kth power residues are only $\frac{1}{2}(n - 1)$ in number they are exactly the same values as the quadratic residues to the modulus n. (See Table 15.) Looking further into this it will be seen that when there is a full complement of $(n - 1)$ residues to any modulus, those of the even indices, $(k = 0, 2, 4, \ldots)$ are quadratic residues and therefore those of the odd indices are non-residues.

Since in these tables we are dealing with residues to the powers of integers—of the index k—this enables us to apply to the multiplication of residues a logarithmic procedure similar to that used in ordinary arithmetic. Thus, to find the product (mod n) of two or more residues it is only necessary to add their corresponding k values together and then find the residue in line with this new k.

The procedure will be illustrated by taking an example from each table. First, from Table 16*a*, if we take two values in the (mod 13) column, say 8 and 12 the product of which is 96. Then $96 \equiv 5 \pmod{13} \equiv 2^9$. The k values corresponding to 8 and 12 are 3 and 6, which added together equal 9.

Similarly, (Table 16*b*) to the modulus 11:

$$3 \times 9 = 27 \equiv 5 \equiv 3^3, \text{ and } 1 + 2 = 3.$$

Again, (Table 16*c*) to the modulus 19:

$$6 \times 17 = 102 \equiv 7 \equiv 10^{12}, \text{ and } 4 + 8 = 12.$$

(6) It is now possible to provide an explanation for the rule given previously concerning the multiplication of quadratic and non-quadratic residues. The addition of two even or two odd k's produces an even k, and hence a quadratic residue, whilst the addition of one even and one odd k is always an odd value and therefore a quadratic non-residue.

Attentive readers will by now, I am sure, have noticed that the sequence of residues (mod 19) to 2^k and 10^k are identical but in reverse; how many, I wonder, have associated this sequence with that described in earlier chapters (*a*) for determining whether a given number is divisible by 19, and (*b*) as part of one process for constructing the recurring decimal of 1/19.

EXERCISES

1. Show that 101 is a divisor of 99999999.
2. Show that $2^{22} - 1$ is divisible by 23.
3. The following five numbers all have the correct 'endings' and digital roots of square numbers. Given that the quadratic residues of 7 and 11 are respectively (0, 1, 2, 4) and (0, 1, 3, 4, 5, 9) find two of the five which cannot be square.

a	2544025
b	2908456
c	2712609
d	2893401
e	2354464

12 FACTORISATION

(1) A large part of that branch of mathematics somewhat loosely termed 'Number Theory' is either devoted to, or has its origins in the quest for prime numbers. The determination of whether a given (large) number is prime, or if not what are its prime factors, is frequently a matter of interest and more often than not one of considerable difficulty. In this chapter we shall examine some of the methods which have been developed for reducing these difficulties.

Large numbers or small, there are of course some elementary observations which can be made at once. Even numbers and multiples of 5 are recognisable at sight, multiples of 3 respond to a simple and rapid mental check and thanks to a lucky combination of factors those of 7, 11, 13 and 37 are found with little effort.

As we have seen in an earlier chapter many other of the smaller primes can be eliminated without having to use the laborious process of testing by actual division. These methods obviously dispose of vast quantities of the natural numbers—even numbers and multiples of 3 alone take care of two-thirds of them, for instance—but there remains an infinity of integers which are either prime or have larger prime factors than the methods used in chapters 1 and 2 are equipped to deal with. These can be divided broadly into two classes; in the one case numbers of a particular structure such as, say, those expressed generally by the formula $x^n \pm 1$ for which the 'form' of possible factors can usually be determined fairly easily. On the other hand there are amorphous numbers, or shall we say numbers of no known history, whose factor forms can only be found by congruence techniques which for numbers of six or seven digits are inferior to other methods of factorisation and are quite impracticable for larger numbers.* Incidentally, it must be noted that in testing a given number N for possible prime factors it is not necessary to

* See *Advanced Algebra*, Barnard & Child, pp. 192–3.

try a divisor greater than $\sqrt{N}$ since if there is a factor larger than this there must also be a smaller one which has already been disclosed.

Since testing by direct division by the primes in sequence is apparently only really applicable to numbers already well covered by published factor tables it would seem that there is little point in pursuing this method further. As we shall see later, however, in one of the most successful methods of factorisation it is of the utmost importance to determine the highest possible limit of primes which are *not* factors of the number we are trying to crack. Apart from this it will be obvious that if we have eliminated all the primes less than the cube root of a given number N then there can be at most two factors of N and one of these must lie between $\sqrt[3]{N}$ and $\sqrt{N}$, clearly a piece of valuable information.

(2) The best way of testing a number of small prime divisors 'en bloc' is to employ Euclid's Algorithm for finding the H.C.F. (highest common factor) of two numbers. In essence this depends upon the fact that if two numbers have a factor in common then the remainder after dividing one by the other will also contain this factor. Similarly, on dividing this remainder into the previous divisor if there is a further remainder this will also contain the factor, and so on. Continuing in this manner the position is finally arrived at where either the remainder r_n equals 1, in which case both numbers are relatively prime, or r_{n-1} is a multiple of r_n. In the latter event the integer r_n is the greatest common factor of the two numbers.

As so often happens in arithmetic the method is far simpler to operate than to describe and an elementary example should clarify the above explanation. Taking the numbers 21 and 56, we proceed to divide thus:

$$56/21 = 2, + 14; 21/14 = 1, + 7; 14/7 = 2 \text{ exactly,}$$

and therefore the last divisor—7—is the H.C.F. of the two numbers 21 and 56.

It will be obvious then from the above that if we prepare a composite number containing as factors all the prime divisors we wish to test as possible factors of a given number N, Euclid's Algorithm will supply the answer. It is conventional to express the product of successive primes by the Greek capital Π, and specifically for instance the product of 3 . 7 . 11 . 13 . . . . 101 is written $\underset{3\leqslant p\leqslant 101}{\Pi(p)}$. (For

obvious reasons 5 need not usually be included.) The following example will show the basic principles of this method of factorisation.

Let $N = 943.$

Now 943 is clearly not a multiple of 3, nor yet of 7, 11, or 13. And since it is less than 31^2 its only possible factors are then 17, 19, 23, or 29. The product of these four primes is 215441. The algorithm then proceeds;

```
               943)215441
                   1886
                    2684
                    1886
                     7981
                     7544
Remainder =           437)943
                          874
Remainder =                69)437
                              414
Remainder =                    23)69
                                  69
                                   0
```

And therefore 23 is a factor of 943.

The following illustration is given, not with the intention of boring the reader, who can easily skip it if he feels inclined, but in order to give those interested some feel of the 'weight' of the calculation for numbers of a moderate size. The 'time' advantage over trial divisions by the separate primes increases with the size of N.

We shall examine the number $N = 20{,}699{,}411$, and since the primes 3, 7, 11, 13, and 37 can be so readily eliminated it will be assumed that they have been already tested and we then formulate

$$\underset{17\leqslant p\leqslant 53}{\Pi(p)} \quad (\text{ex } 37) = 29330271959743.$$

The algorithm then proceeds:

```
20699411/29330271959743
         20699411
         86308609
         82797644
          35109655
          20699411
          144102449
          124196466
           199059837
           186294699
            127651384
            124196466
              34549183
              20699411
              13849772/20699411
                       13849772
                        6849639/13849772
                                13699278
                                  150494/6849639
                                         601976
                                          829879
                                          752470
                                           77409/150494
                                                  77409
                                                  73085
                   73085/77409
                         73085
                          4324/73085
                               4324
                               29843
                               25944
                                3901/4324
                                     3901
                                      423/3901
                                          3807
                                            94/423
                                               376
                                                47/94
                                                   94
                                                    0
```


Therefore 47 is a factor of N.

Dividing out, we have $N = 47 \times 440413$

To continue with the next few possible prime divisors we take

$$59 \,.\, 61 \,.\, 67 \,.\, \ldots \,.\, 83 = \prod_{59 \leqslant p \leqslant 83}(p) = 8472681192817.$$

```
440413/8472681192817
       440413
       4068551
       3963717
        1048341
         880826
         1675159
         1321239
          3539202
          3523304
           1589881
           1321239
            2686427
            2642478
              43949/440413
                    43949
                      923/43949
                          3692
                           7029
                           6461
                            568/923
                                568
                                355/568
                                    355
                                    213/355
                                        213
                                        142/213
                                            142
                                             71/142
                                                142
                                                  0
```

Therefore 71 is a factor of 440413.

Thus $N = 20699411 = 47 \times 71 \times 6203$. Moreover since $\sqrt{6203}$ is less than 80 and all primes up to and including 83 have been tested then 6203 is prime and we have the complete factorisation.

For dealing with large numbers the time consumed by this procedure is relatively trivial and it is generally preferable to employ stages of much larger $\Pi(p)$s.

To save some duplication of effort I give below the products of primes which I have found most convenient in practice.

Table 17

Products of Successive Primes $\Pi(p)$

	$\Pi(p)$
$17 \leqslant p \leqslant 89$ (ex 37)	2139188909860009429367977752 1
$97 \leqslant p \leqslant 173$	70087808817907376609189021533229 41
$179 \leqslant p \leqslant 251$	3857756650124091541408823809774 13
$257 \leqslant p \leqslant 337$	2310505354842804355389431964226253 93
$347 \leqslant p \leqslant 419$	323637432688764181699774272149482 1
$421 \leqslant p \leqslant 499$	178617233785167153407367386470071612 71

There is no need to be intimidated by the size of these numbers; the first division may take a little time but after the first remainder is found the rest of the algorithm requires only a matter of minutes. Generally speaking a number of about ten or twelve digits can be tested for all the prime divisors less than 500 in under the hour and if a hand or desk calculator is available this time will be considerably reduced.

Naturally if we are dealing with a number whose only possible factors are of a particular 'form' it is only necessary to use primes of this form in preparing multiple products for use in the algorithm. This means that either the time involved can be cut or the tests can be carried to much higher values of p.

(3) We turn now to a method of factorisation which is infallible, at least in theory. The qualification must be made because there are, of course, random numbers of such magnitude that the labour involved in any computational method is prohibitive.

This method is practicable for numbers of up to about twenty digits, although it must be admitted that if the number turns out to

be prime, as in the case of I_{19}, the amount of man-hours required for the test calls for some dedication.

The procedure is based on an identity known to every schoolboy, namely that

$$x^2 - y^2 = (x + y)(x - y), x > y.$$

It is easily shown that all odd numbers can be expressed as the difference of two squares. For if N, an odd number, is equal to ab, then a and b are both odd and $(a + b)$ and $(a - b)$ are therefore even. Then

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = \frac{4ab}{4} = ab = N$$

When N is prime then $x + y = N$, and $x - y = 1$, as seen in the example $13 = (7 + 6)(7 - 6) = 7^2 - 6^2$. Composite numbers can also be expressed in this manner but in addition, in at least one other way. Thus when $N = 105$ we have:

$$\begin{aligned} 105 &= 53^2 - 52^2 = (53 + 52)(53 - 52) = 105 \times 1. \\ &= 19^2 - 16^2 = (19 + 16)(19 - 16) = 35 \times 3. \\ &= 13^2 - 8^2 = (13 + 8)(13 - 8) = 21 \times 5. \\ &= 11^2 - 4^2 = (11 + 4)(11 - 4) = 15 \times 7. \end{aligned}$$

In order then to divide a given odd number N into two factors it is necessary to find two squares which have N as their difference. In practice this is effected by subtracting N in turn from a rising sequence of square numbers until a square turns up in the remainders. To take a simple example, let $N = 217$. The first square greater than 217 is $15^2 = 225$, and we might then proceed as follows:

$15^2 = 225$	$16^2 = 256$	$17^2 = 289$	$18^2 = 324$	$19^2 = 361$
217	217	217	217	217
8	39	72	107	144

Now $144 = 12^2$, and therefore $217 = 19^2 - 12^2 = 31 \times 7$.

This example serves to illustrate the basic principle of the method but the technique is far too clumsy to apply to large numbers. However, a simple manoeuvre enables us to set out the work in a much more compact and workmanlike fashion.

Since $(n + 1)^2 = n^2 + (2n + 1)$, successive squares increase by successive odd number increments and in consequence the differences between the rising squares and N will increase by the

same increments. Thus the above individual subtractions can be replaced by cumulative sums in the following manner.

$$
\begin{array}{rr}
15^2 = & 225 \\
N = & \underline{217} \\
 & 8 \\
(2 \times 15) + 1 = & \underline{31} \\
16^2 - N = & 39 \\
 & \underline{33} \\
17^2 - N & 72 \\
 & \underline{35} \\
\text{etc.} & 107 \\
 & \underline{37} \\
 & 144 = 12^2
\end{array}
$$

(4) The above introduction to this subject will be expanded later, but first it is obviously important to be able to recognise a square when one is found. When dealing with moderately small numbers it is only necessary to have at hand a table of square numbers of which many are available. As N becomes large, however, one must eventually resort to the traditional method of square root extraction. Even so there are some observations to be made which will reduce this necessity to a minimum.

It has already been noted that the digital roots of square numbers—that is, the quadratic residues of 9—can only be 1, 4, 7, and 9 and this is a test which can be applied with speed. Examination of a table of squares will also show that the last two or three digits of square numbers are confined to certain definite patterns. Using the convention that $\underline{0}$ represents any *even* digit and $\underline{1}$, any *odd* digit, it will be seen that the only possible 'square endings' are $\underline{0}1$, $\underline{0}4$, $\underline{1}6$, $\underline{0}9$, 00, and 125, 225, 625. A still closer inspection reveals that a further restriction can put upon numbers with the endings $\underline{0}1$ and $\underline{0}9$, namely that squares in these categories can only have the following forms of endings:

. . . $\underline{0}01$	. . . $\underline{0}09$
. . . $\underline{1}21$	. . . $\underline{1}29$
. . . $\underline{0}41$	. . . $\underline{0}49$
. . . $\underline{1}61$	. . . $\underline{1}69$
. . . $\underline{0}81$	. . . $\underline{0}89$

Numbers satisfying these conditions are still not necessarily square but if the suspect number is large it may be preferable to make further tests by division by small primes rather than resort to the tedious process of root extraction. For instance if on division by 7 the remainder (residue) is not 0, 1, 2, or 4 (see Table 15), the number is not square.

Taken together the above tests will eliminate a considerable number of integers which cannot possibly be squares.

(5) We are now equipped to attempt the factorisation of a moderately large number and I have chosen one of sufficient magnitude to demonstrate both the power of the method and also to leave some elbow room for the development of some useful short-circuiting techniques.

Let, for example, $N = 1{,}002{,}387{,}143$. To begin with the smallest square greater than N is found by the customary square root extraction process, thus:

$$
\begin{array}{lrl}
 & 3\ 1\ 6\ 6\ 1^2 & \\
 & 1002387143 & \\
 & 9 & \\
61 & 102 & \\
 & 61 & \\
626 & 4138 & \\
 & 3756 & \\
6326 & 38271 & \\
 & 37956 & \\
63321 & 31543 & \text{(to ensure that the} \\
 & 63321 & \text{square is larger than} \\
 & -\,31778 & N\text{)}
\end{array}
$$

We now have $N = 31661^2 - 31778$ and the successive differences from the rising sequence of squares are found as follows:
(Note that $2 \times 31661 + 1 = 63323$)

$$
\begin{array}{rl}
31778 & \\
\underline{63323} & \\
95101^* & = 31662^2 - N \\
\underline{63325} & \\
158426 & \\
\underline{63327} & \\
221753 & \\
\underline{63329} & \\
285082 & \\
\underline{63331} & \\
348413 & \\
\underline{63333} & \\
411746 & \\
\underline{63335} & \\
475081^* & = 31668^2 - N \\
\underline{63337} & \\
538418 & \\
\underline{63339} & \\
601757 & \\
\underline{63341} & \\
665098 & \\
\underline{63343} & \\
728441^* & = 31672^2 - N \\
\underline{63345} & \\
791786 & \\
\text{etc.} &
\end{array}
$$

It will be seen (the reader can confirm this by continuing the column) that after ten steps the final digits of the sums begin to repeat themselves in the same order, and that in addition the penultimate digits have the same parity as their predecessors. Furthermore we note that in the intervals of ten steps only two sums have the two terminal digits of a possible square*. These correspond with the numbers $31662^2 - N$, and $31668^2 - N$ and are 95101 and 475081 respectively.

Owing to the cyclic pattern of the 'endings' it follows that the only sums in which squares can possibly appear are those related to the squares 31662^2, 31672^2, $31682^2, \ldots$ and 31668^2, $31678^2, \ldots$

etc. We can therefore dispense with the intermediate steps and set up two columns, each of which will advance at ten times the rate we started with.

This presents no difficulties since we are using here an Arithmetical Progression and the total sum of ten such steps is readily found by coupling them in five equal pairs, thus:

$$
\begin{array}{rrrl}
63325 & 63327 & 63329 & \\
\underline{63343} & \underline{63341} & \underline{63339} & \\
126668 & 126668 & 126668 & \text{etc. and } 5 \times 126668 = 633340.
\end{array}
$$

(Note. Each successive 'ten steps' increment increases by 200, and in fact it is easily shown by A.P. formulae that when, as in this case, the common difference per 'step' is 2, then the incremental increase for n steps is $2n^2$. Thus the increase for one step is 2, for 10 steps—200, 20 steps—800, 90 steps—16200, and so on.)

To proceed we now set up the following columns:

$$
\begin{array}{rlrl}
95101 & = 31662^2 - N & {*}\ 475081 & = 31668^2 - N \\
\underline{633340} & & \underline{633460} & \\
{*}\ 728441 & = 31672^2 - N & 1108541 & = 31678^2 - N \\
\underline{633540} & & \underline{633660} & \\
1361981 & \text{etc.} & {*}1742201 & \text{etc.} \\
\underline{633740} & & \underline{633860} & \\
{*}1995721 & & 2376061 & \\
\underline{633940} & & \underline{634060} & \\
2629661 & & {*}3010121 &
\end{array}
$$

None of these sums are squares but we now note that whilst they all have the required last two digits of squares only those starred have the necessary parity for the third digit from the last. Successive A.P. numbers are always rigid pattern-followers and if the above columns are extended it will still be found that only every other sum has the three permissible terminal digits of a possible square.

It is therefore possible to reduce the number of successive additions still further, this time to cover twenty of the original steps. As the following columns will now have all the correct terminal digits of squares we shall pay attention to the digital roots in the next summations. Starting with the starred members of the last columns we then proceed:

	D.R.		D.R.
$728441 = 31672^2 - N$	8	$475081 = 31668^2 - N$	7
1267280		1267120	
$1995721 = 31692^2 - N$	7	1742201	8
1268080		1267920	
3263801	5	3010121	8
1268880		1268720	
4532681	2	$4278841 = 31728^2 - N$	7
1269680		1269520	
$5802361 = 31752^2 - N$	7	5548361	5
		1270320	
		6818681	2
		1271120	
		$8089801 = 31788^2 - N$	7

Note that as each of the above additions is equivalent to twenty of the original steps the increments increase each time by 800. It will also be seen that the only digital root of a square to appear in this cycle—namely 7—occurs at every third sum (i.e. at every sixty of the original steps). Consequently there is no point in looking for squares in the sums of anything but the $(31692 + 60k)^2 - N$, and $(31668 + 60k)^2 - N$ cycles, and as no squares have yet appeared we now set up two further columns which in addition to having the required terminal digits will contain only sums possessing the digital root of 7. (Incidentally, since each of these sums will now correspond to 60 of the original steps each addition increases by $2 \times 60^2 = 7200$.)

$1995721 = 31692^2 - N$	$4278841 = 31728^2 - N$
3806640	3810960
$5802361 = 31752^2 - N$	$8089801 = 31788^2 - N$
3813840	
$9616201 = 31812^2 - N$	

We now have a square at last, for $9616201 = 3101^2$.

Hence $N = 1002387143 = 31812^2 - 3101^2$
$= (31812 + 3101)(31812 - 3101)$
$= 34913 \times 28711$. Which is the complete factorisation.

Shorn of explanatory verbiage and repetitions in the columns the above process can be seen to be remarkably economical in both time and paper: it is still open to further refinement as will be shown later. At the same time modifications of the technique may sometimes be required; with some Ns, for instance, the (square) digital roots are not evenly spaced and this entails setting up four columns at this stage. There should be no difficulty, however, in extemporising to meet any situation if the above arguments have been thoroughly mastered.

(6) A fundamental difficulty arises in this method of factorisation when N has only two factors which have widely different orders of magnitude. When this happens, or in the extreme case when N is both large and prime, the operation becomes prolonged, perhaps to an intolerable extent.

If the process then is to remain practicable in such circumstances it is essential to find some way of at least reducing this difficulty.

Now it will be clear that however many factors N may possess the method described above produces only two; one or both of these may of course be composite.

Let $N = ab$, and as we want N in the form $x^2 - y^2$ then we must have $a = x + y$, and $b = x - y$.

(It should be noted that since N is odd—or can be made so by division by some power of 2—then both $x + y$ and $x - y$ must also be odd.)

If a and b are of the same order of magnitude we can say that $a \sim b \sim \sqrt{N}$, which implies that y is relatively small and will be found early in the proceedings. On the other hand if a and b differ appreciably then it can be said that $a =$ (approximately) kb, where k is an integer. Obviously if k can be found then kN will have the two nearly equal factors a and kb which will quickly be disclosed.

Unfortunately there is no way of predicting the value of k and so it can only be found by trial. That is, if the above algorithm shows no signs of finding a square it should be abandoned and a new test started on $3N$, then on $5N$, and so on. (The reason why even values of k are not recommended can be explained as follows. When k is odd then both $x + y$ and $x - y$ must be odd and the condition noted above is satisfied. When, however a is nearly equal say to $2b$ and the algorithm is applied to $2N$ no solution will

emerge because we have now made one of the factors even whilst the other remains odd and it is no longer possible for them to assume the forms $x + y$ and $x - y$. Furthermore, if $4N$ is tried, although now both new factors will be even they will be in the same ratio as before and no advantage will have been gained. In such a case, therefore, N must be multiplied by 8 before the new factors become nearly equal, and in general if k is an even number N must be multiplied by $4k$.)

To recapitulate, it has been shown that if $N = ab$ and $a \sim kb$, then the difference of squares method will quickly produce the factors of N if it is applied to

kN when k is an odd number,
$4kN$ when k is an even number.

It may come as a surprise that increasing the size of the number to be tested actually leads to a quicker solution of the problem but the reader can easily satisfy himself of the truth of this.

It will now be seen why it is important to eliminate as many of the smaller prime divisors as possible before beginning this procedure since we are then able to define the lowest limit of k. Thus when it is clear that there are no factors less than, say W, the first k to be tried should be just greater than N/W^2.

(7) The foregoing process is powerful and yet requires little more than an attention to detail which one must accept in this brand of mathematics. After a few practice shots at numbers of say, three to six digits—Premium Bond numbers provide excellent material—the reader should be able to tackle quite large numbers with confidence. In fact he will be equipped to perform what some writers have regarded as near-miracles.

On the subject of primes, and referring to an often quoted Fermat legend (repeated even as recently as March 1964 in *Scientific American* in almost identical terms), Kasner and Newman in their stimulating book *Mathematics and the Imagination* have this to say (p. 187):

> 'Curiously enough there is reason to believe that certain mathematicians of the seventeenth century, who spent a great deal of time on number theory, had means of recognising primes unknown to us, . . . it is still a source of wonder that Fermat replied without a moments hesitation to a letter which asked whether 100895598169 was a prime, that it was the product of 898423 and 112303, and that each of these numbers was prime. Without a general formula for all primes, a mathematician, even today, might spend years hunting for the correct answer.'

The 'difference of squares' method, which was of course well known to Fermat, breaks this number down almost before we can get started, but look what happens when we apply the process to $8N$. I give the calculation in full in order to show that the time involved is trivial.

We have $N = 1008955981169$; $8N = 807164785352$. Then extracting the square root of $8N$:

	807164785352(898424
	64
169	1671
	1521
1788	15064
	14304
17964	76078
	71856
179682	422253
	359364
1796844	6288952
	7187376
	− 898424

Thus
$$\begin{aligned} 8N &= 898424^2 - 898424 \\ &= 898424\,(898424 - 1) \\ &= 8 \times 112303 \times 898423 \end{aligned}$$

Therefore
$$N = 112303 \times 898423.$$

So much for 'lost methods', 'general formulae for primes' and 'years of hunting'.

(8) Before ending this chapter it will perhaps be appropriate to include a note on 'checking the work'. In calculations of this kind, which may on occasion become protracted, it is always well at intervals to check the accuracy and the progress of the operation.

This is easily carried out and it will be enough to give only a simple example.

Suppose the 'difference of squares' algorithm is being applied to $N = 3527$, a number which is easily seen to have none of the factors 3, 7, 11, and 13. The working starts with

$$N = 60^2 - 73$$

and proceeds to the point

$$N = 126^2 - 12349$$

without a square number appearing. At this stage it is decided to make a check.

The largest square less than 12349 is $111^2 = 12321$ which differs from 12349 by 28.

Now:

$$126^2 - 111^2 = 15 \times 237 = 3555,$$

And

$$3555 - N = 28.$$

Therefore the calculations are without error and furthermore the check shows that all the primes between $\sqrt{N}$ and 15 have been tested. It follows that 3527 must therefore be prime.

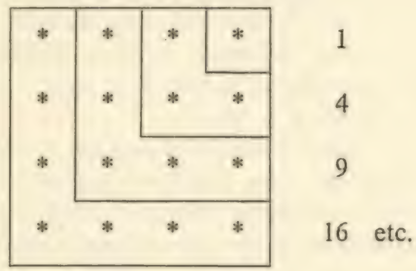
This method of checking should never be neglected since it not only confirms the accuracy of the work but at the same time provides an indication of when one should start again with a new value of kN.

EXERCISES

1. Find one factor common to both 16733 and 35699.
2. $N = 1181027$ has one factor less than 30. What is it?
3. Using the 'difference of squares' method find the factors of $N = 1093709$.

13 MORE FACTORISATION METHODS

(1) We are all familiar with the 'square' numbers, which can be represented graphically in the form:



'Triangular' numbers are those formed by the cumulative sums of the rows in a triangular array, thus:

```
        *          1
      *   *        3
    *   *   *      6
  *   *   *   *    10
*   *   *   *   *  15  etc.
```

and they produce a sequence of which the nth term is

$$\frac{n(n+1)}{2}.$$

These numbers share with the squares many useful properties, not all of which seem to have received full recognition.

In my opinion their employment in factorisation techniques is vastly to be preferred to that of the square numbers. It is easily seen that, just as with the squares, any integer which can be expressed as the difference of two triangular numbers is composite and, when so resolved, the factors can be found immediately.

Thus, $(x > y)$:

$$\frac{x(x+1)}{2} - \frac{y(y+1)}{2} = \frac{x^2 - y^2 + x - y}{2}$$

$$= \frac{(x-y)(x+y) + (x-y)}{2}$$

$$= \frac{(x-y)(x+y+1)}{2}$$

When using these numbers for factorisation the same use can be made of digital roots, terminal digits and multiples of N as those described in some detail in the last chapter, with, of course due recognition of the basic difference in these properties. Thus, triangular numbers have digital roots of 1, 3, 6, and 9, and can only end in 0, 1, 03, 53, 5, 6, 28, 78.

In this system the 'step' increments increase by 1, instead of 2, but the digital roots, etc., have exactly similar cyclic properties and offer the same opportunities for abbreviation. The method has one disadvantage, namely that tables of triangular numbers are not so readily available as those of squares. They are, however, easily constructed particularly if one has access to a hand or desk calculating device, and in any case the loss is not all that important when dealing with large numbers. A suspected triangular number, that is one having the right ending and digital root, is tested by first multiplying by two and then extracting the square root. Thus: $T = 11935$. Then $11935 \times 2 = 23870$.

$$\begin{array}{rrl}
 & 1\ 5\ 4^2 & \\
2T = & 23870 & = 154^2 + 154 = 154 \times 155 \\
 & \underline{1} & \text{and therefore } T = (154 \times 155)/2 \\
25 & 138 & \text{which is of triangular form} \\
 & \underline{125} & \\
304 & 1370 & \\
 & \underline{1216} & \\
 & 154 &
\end{array}$$

After what has been said there should now be no difficulty in understanding the working of the following example.

To find the factors of $N = 437891$ $(= 397 \times 1103)$.

First find the least triangular number $>N$ by extracting the square root of $2N$.

$$\begin{array}{rr}
 & 9\ 3\ 6^2 \\
2N = & 875782 \\
 & \underline{81} \\
183 & 657 \\
 & \underline{549} \\
1866 & 10882 \\
 & 11196
\end{array}$$

and we have $\dfrac{936 \times 937}{2} = 438516$. $(=$ say, $936_t)$

The calculation then continues:

$$\begin{array}{rrl}
 & 438516 & \\
-N & 437891 & \\
\hline
 & 625 & = 936_t - N \\
 & 937 & \\
\hline
 & 1562 & \\
 & 938 & \\
\hline
 & 2500 & \\
 & 939 & \\
\hline
 & 3439 & \\
 & 940 & \\
\hline
 & 4379 & (940_t - N) \\
 & 941 & \\
\hline
 & 5320 & \\
 & 942 & \\
\hline
 & 6262 & \\
 & 943 & \\
\hline
 & 7205 & \\
 & 944 & \\
\hline
 & 8149 & \\
 & 945 & \\
\hline
 & 9094 & \\
 & 946 & \\
\hline
 & 10040 & \\
 & 947 & \\
\hline
 & 10987 & \\
 & 948 & \\
\hline
 & 11935 & = (948_t - N) = 154_t
\end{array}$$

We now have, $N = 948_t - 154_t$ and therefore N is composite, its factors being found as follows:

$$\begin{array}{rr}
 & 948 \\
+ & 155 \\
\hline
 & 1103
\end{array}
\qquad
\begin{array}{rr}
 & 948 \\
- & 154 \\
\hline
 & 794 = 2 \times 397
\end{array}$$

It will be noticed that as each value of $(x_t - N)$ is found x is numerically equal to the previous incremental addition; because of

this it is simpler to keep track of the operation than in the difference of squares method. The greatest advantage that the triangular numbers have over the squares in this system of factorisation lies in the number of steps required for ultimate solution. In the above example the factors were found after only twelve steps whereas the 'difference of squares' process requires 87 steps for the same N.

(2) We turn now to some less orthodox systems which can be used for finding factors. They are included more for their individual interest than for their practical value in attacking large numbers, although any of them might well repay some development work.

The first is simply a variant on the method described in the last chapter. It is deliberately being treated separately because unlike the former method its power is seen to the best advantage when N is either prime or has factors of widely different orders of magnitude—that is, when k proves to be large.

The principle becomes clear when it is remembered that whilst there are exactly $p - 1$ quadratic residues, all of them different, to any prime p, the number of residues to a composite number M is always less than $M - 1$ and some of them are repeated at least once.

Briefly, the process starts with a progression of steps produced in exactly the same way as in the 'difference of squares' method, but in this case the progressive totals are not allowed to exceed N in value. When this occurs N is subtracted and the steps are then continued in the normal way. From this point onward a watch is then kept for any sum which has appeared before and as soon as a duplication is seen the factorisation can be completed.

To give an example of the process, we have:

$$N = 2533 = 51^2 - 68. \qquad (= 17 \times 149. \ \ k \sim 9)$$

Then,

68	1311	118
103	125	145
171	1436	263
105	127	147
276	1563	410
107	129	149
383	1692 (65^2)	559 (75^2)
109	131	151
492 (55^2)	1823 (66^2)*	710
111	133	153
603	1956	863
113	135	155
716	2091	1018
115	137	157
831	2228	1175
117	139	159
948	2367	1334 (80^2)
119	141	161
1067 (60^2)	2508	1495
121	143	163
1188	2651 $> N$	1658
123	2533	165
1311	118	1823 (83^2)*

* The sum 1823 has been repeated and the factors are found thus:

$$N = 66^2 - 1823$$
$$2N = 83^2 - 1823$$

Subtracting $\quad N = 83^2 - 66^2 = (83 + 66)(83 - 66)$

And then $\quad N = 149 \times 17.$

As a suggestion for further research into this subject, it might be of interest to note that the residues to triangular numbers have similar properties to quadratic residues and are also duplicated, sometimes at frequent intervals, when N is composite.

(3) The following algorithm is not so much a method of factorisation as a device for reducing the size of the number under test.

I have not seen it mentioned before but that, of course, does not give it any claim to originality.

To find the factors of N first set up a column (A) in which the first term is $(N - 1)/2$, the following lines being successively reduced by unity. At the same time odd integers 1, 3, 5, 7, . . . are set alongside these terms in a column (B). Then those numbers in col. B which divide their companions in col. A are also divisors of N.

To illustrate, we take $N = 1001$. Then $(N - 1)/2 = 500$.

Then proceed,

A	B
500	1
499	3
498	5
497	7

And since $497/7 = 71$, then 7 is a factor of 1001. Furthermore as $(2 \times 71) + 1 = 143 = 11 \times 13$, we have the other factors. Or alternatively the columns could be continued thus:

496	9
495	11 ($495 = 11 \times 45$)
494	13 ($494 = 13 \times 38$)

Since the integers in col. A decrease by exactly half the increase of those in col. B it is a simple matter to arrange for composite numbers to be omitted from col. B, or to skip as many numbers as we like. In the first instance column B can, and indeed should, contain only the odd primes, and in the second whenever the possible factors of N are known to be of a particular 'form' only primes of this form need be tabulated.

Thus, for example if $N = 11111$, and it is known, as we saw earlier, that only numbers of the form $30n + 1$ and $30n + 11$ are eligible then we can proceed as follows:

$$(N - 1)/2 = 5555.$$

Then we have

5555	1
5550	11
5540	31
5535	41

On testing it is found that $5535/41 = 135$.

And since $(135 \times 2) + 1 = 271$, we have the factors 41 and 271.

This method has the following advantages: it allows little opportunity for errors in calculation; with the aid of factor tables it is effective in quickly eliminating quantities of small primes particularly when they are of known 'form'; and as it starts with $N/2$ it therefore virtually doubles the range of the available factor tables.

(4) The method of factorisation about to be described is taken from L. E. Dickson's *History of the Theory of Numbers* Vol. 1 and therein is attributed to D. Biddle (1911).

It is included here not for its practicability but to provide a basis for speculation on the devious processes which led up to its conception.

Process. Express N as $S^2 + A$ where S^2 is the largest square $<N$. Write three rows of numbers, the first beginning with A, (or $A - S$ if $A > S$); the second beginning with S (or $S + 1$, in the latter case), and increasing by 1, the third beginning with S and decreasing by 1.

Let A_n, B_n, C_n be the nth elements in the respective rows. Then $C_n = C_{n-1} - 1$; $B_n = B_{n-1} + 1$; $A_n = A_{n-1} + B_{n-1} - kC_n$. When $(A_{n-1} + B_{n-1})$ is greater than C_n, a multiple of C_n is subtracted so as to leave a positive remainder and then $B_n = B_{n-1} + k$. When a value of n is reached for which $A_n = 0$, then $N = B_n \times C_n$.

For an example we take $N = 589 = 24^2 + 13$

A	13	14	17	1	9	0
B	24	25	26	28	29	31
C	24	23	22	21	20	19

The factors of $N = 589$ are thus disclosed as 19 and 31.

I am sorry if the above account lacks a certain amount of elegance but am comforted by Dickson's laconic note 'It may be best to start with $2N$.'

EXERCISES

1. Find the divisors of the following numbers using *any* method of factorisation.

a	15871
b	15853
c	15863

2. The factors of $N = 13333$ are of the form $22k + 1$. What are they?
3. The factors of $N = 548497$ are of the form $26k + 1$. What are they?

14 A NOTE ON THE CONVERSE OF FERMAT'S THEOREM

(1) Many readers must have been puzzled on reading that such and such a (large) number is composite but its factors are not known. One would think that the authors of books in which such statements appear—and there are quite a few of them in the 'popular' mathematics field—might have spared a few lines to explain the basic principle on which an observation of this sort depends.

Let us consider Fermat's theorem again. This states in effect, that if p is a prime and a is neither equal to, nor a multiple of p, then $(a^{p-1} - 1)$ is a multiple of p. Or in congruence language $a^{p-1} \equiv 1 \pmod p$. All this is perfectly true but one must be careful not to infer that the converse is necessarily true, namely that if $a^{N-1} \equiv 1 \pmod N$ then N must be prime. The probability that it is, is very high but in spite of this it can be proved that there are an infinity of composite N's which satisfy the congruence.

Lucas first laid down, and in 1891 proved the necessary conditions for a true converse of the theorem, namely that 'If $a^x \equiv 1 \pmod N$ for $x = N - 1$, but *not* for x a proper divisor of $N - 1$, then N is a prime.'

Since then Lehmer in particular has developed many refinements to the theory, so much so that it can be considered now as one of the few valid tests for the primality of large numbers. The arguments are somewhat advanced and quite beyond the scope of this book; probably the best presentation is to be found in D. H. Lehmer's paper, 'Tests for primality by the converse of Fermat's theorem', *Bull. Amer. Math. Soc.*, vol. 33, 1927, pp. 327–340.

On the other hand the converse presents no difficulties when applied as a test for *composite* numbers, and it is legitimate to conclude that if $a^{N-1} \equiv r \pmod N$, where r proves to have any other value than 1, then N is *not* prime and is therefore a product of two or more primes.

To carry out this test is is necessary to find the remainder on

dividing a^{N-1} by N and as a can take any value prime to N it is convenient to let $a = 2$. The actual calculation uses the following properties of a congruence, namely that if $2 \equiv R \pmod N$ then $2^2 \equiv R^2, 2^3 \equiv 2R^2$, etc. It is advisable to set out the operations in an orderly manner and the following explanation and the three worked examples—although applied to only small N's—will indicate the procedure to be followed in testing numbers of any magnitude. The calculation employs two columns of figures labelled here A and $2^A \pmod N$ respectively. The left hand column (A) is constructed first and starts with $N - 1$. Each successive line is obtained by dividing the previous one by 2, (after subtracting 1 if the number is odd). The last entry is, of course 1 and against this the next column is started from the bottom with the number 2. (It is as well at this stage to make a mark against the odd numbers in Col. A as a reminder in what follows.)

The rising sequence of this second column is now formed by successive squaring, the resultant squares also being doubled where the corresponding entries in Col. A are odd. Each value is then divided by N and the remainder entered in the next line above. The final entry at the top of the column is then the residue (mod N) of 2^{N-1} and if this is any number other than 1 then N is composite.

Example 1

$$N = 437$$

A	$2^A \pmod N$	
436	359	
218	213	
109*	173	
54	58	etc.
27*	170	$2 \times 111^2 = 24642 \equiv 170 \pmod{437}$
13*	−111	$2 \times 64^2 = 8192 \equiv 326 \pmod{437}$ see below
6	64	8^2
3*	8	2×2^2
1*	2	

Therefore $2^{437-1} \equiv 359 \pmod{437}$ and N is composite. It should be noted that it is sometimes more convenient to use a

negative value as the remainder since this still becomes positive again after squaring. Thus $2 \times 64^2 = 8192 \equiv 326 \pmod{437}$ and $326 - 437 = -111$. It is simpler to square -111 than 326, a saving of time which can be significant when the modulus is large.

Example 2

$$N = 347$$

A	$2^A \pmod N$
346	1
173*	−1
86	120
43*	193
21*	231
10	−17
5*	32
2	4
1	2

Therefore 347 *may* be prime. (In fact it is.)

Example 3

$$N = 561$$

A	$2^A \pmod N$
560	1
280	1
140	67
70	166
35*	263
17*	—202
8	256
4	16
2	4
1	2

Although it is seen that $2^{560} \equiv 1 \pmod{561}$, 561 is *not* prime. It is in fact equal to $3 \times 11 \times 17$.

There is, of course, no need to confine the value of a to 2, though for obvious reasons it is best to keep to small values. If, for any reason one chooses to use $a = 3$, then the base of the right hand column is started with 3, otherwise the procedure follows exactly the same pattern as with $a = 2$.

Again, there is no need, as can be seen in the above examples, to continue the col. A downwards beyond the point where a^A becomes less than N provided suitable power tables are available. (See Appendix.)

15 CONCLUSION

(1) In what has gone before I have introduced to the reader a very small section of that branch of mathematics known as the Theory of Numbers. The main theme of the book has been factorisation, a subject which, once the bug has bitten is intensely fascinating and yet at the same time embedded in difficulties. I have hoped to give some indication of how some of these difficulties may be chipped away and to have stimulated an interest in research into the subject.

Let no one be deterred from this by the advent of modern electronic computers and their spectacular success in finding ever larger primes of a particular 'form'; the computer may take only minutes to provide the answer to a given problem but its actual programming may well have required hundreds or even thousands of skilled man-hours to prepare. Nor should we forget that most of these programmes depend for their success upon principles laboriously hammered out and tested by mathematical enthusiasts throughout the ages.

A chapter on some of the known properties of primes was almost essential in this context. They have worried mathematicians, particularly amateurs who presumably have more time to spend on matters of doubtful outcome, since their 'separateness' was first recognised. It may be that the modification to Euler's criterion suggested in this chapter (perhaps with further necessary restrictions), could lead somewhere, in which case it ought not to be unduly difficult to extend the test to numbers of the other forms. On the other hand we may be asking altogether the wrong kind of questions about primes. Perhaps an intensive study of the more general 'relative primes' might in the long run be more profitable. Here indeed there could be scope for a new Einstein.

(2) Of course mathematicians, even Number Theory addicts, do not confine their attentions all the time to such fundamental problems

and it might be fitting to close with a few samples of the many curious observations that have been made from time to time.

1. A property of 24; This is the largest value of n that is divisible by all the integers less than $\sqrt{n}$.

2. A property of 30; When $n = 2, 3, 4, 6, 8, 12, 18, 24, 30$, the integers less than, and relatively prime to n are all unity and primes. No number greater than 30 has this property. (Uspensky & Haslet.)

3. In the following subtractions all the original digits are repeated in the answers once only.

$$\begin{aligned} 9876543210 - 0123456789 &= 9753086421 \\ 987654321 - 123456789 &= 864197532 \\ 98754210 - 01245789 &= 97508421 \end{aligned}$$

4. $54 + 72 + 90 = 6^3$

$$54^3 + 72^3 + 90^3 = 108^3 = (72 + 90 - 54)^3$$

5. All the integers except the powers of 2 can be expressed as the sum of consecutive numbers.

6. The addition of unity to the product of any four consecutive numbers produces a square number.

7. In the number 312132 we have, 1 digit between the 1s,
   2 digits between the 2s,
   3 digits between the 3s.

   Can this system be extended?

8. Some examples of multiplications in which all the digits are used once only:

$$7 \times 9403 = 65821: \quad 3 \times 1458 = 6 \times 729.$$

9. 'Fermat's Quotient', $(2^{p-1} - 1)/p$ is only a square when

$$p = 3 \text{ or } 7.$$

10. $512 = (5 + 1 + 2)^3$:

$$47045881000000 = (47 + 4 + 58 + 81)^6.$$

11. $3^3 + 4^3 + 5^3 = 6^3$
$1^3 + 3^3 + 4^3 + 5^3 + 8^3 = 9^3$
$3^3 + 4^3 + 5^3 + 8^3 + 10^3 = 12^3$
$1^3 + 5^3 + 6^3 + 7^3 + 8^3 + 10^3 = 13^3$
$2^3 + 3^3 + 5^3 + 7^3 + 8^3 + 9^3 + 10^3 = 14^3$

12. $11^3 + 12^3 + 13^3 + 14^3 = 20^3$
$6^3 + 7^3 + 8^3 + \ldots + 68^3 + 69^3 = 180^3$
$1134^3 + 1135^3 + 1136^3 + \ldots + 2132^3 + 2133^3 = 16830^3$

13. $4^5 + 5^5 + 6^5 + 7^5 + 9^5 + 11^5 = 12^5$
$5^5 + 10^5 + 11^5 + 16^5 + 19^5 + 29^5 = 30^5$

14. $4! = 24$
$5! = 120$
$7! = 5040$ All become squares on adding 1.
No similar cases are known below 1020!.

15. $6! \times 7! = 10!$

16. The two numbers 57321 and 60984 together contain the ten digits. Each of the squares of these numbers; 3285697041 and 3719048256 contain all ten digits.

17. (*a*) $1 + 2 = 3$
$4 + 5 + 6 = 7 + 8$
$9 + 10 + 11 + 12 = 13 + 14 + 15$ etc.

The highest L.H. number $= n(n + 1)$

(*b*) $3^2 + 4^2 = 5^2$
$10^2 + 11^2 + 12^2 = 13^2 + 14^2$
$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$ etc.

The highest L.H. number $= (2n(n + 1))^2$
But we cannot continue on these lines for

$$5^3 + 6^3 = 7^3 - 2.$$

18. Factors of 10^n containing no 0's. Are any to be found between

$10^2 = 4 \times 25$
$10^3 = 8 \times 125$ and
$10^{33} = 8589934592 \times 116415321826934814453125$?

19. Can the following sequence be continued, 1, 3, 8, 120, . . .? (Where the product of any two integers is $x^2 - 1$.)

APPENDIX

Square Roots

(To seventeen significant figures)

n	$\sqrt{n}$
2	1·4142135623730955
3	1·7320508075688773
5	2·2360679774981797
6	2·4494897427831781
7	2·6457513110645906
8	2·8284271247461909
10	3·1622776601683793
11	3·3166247903553998
13	3·6055512754639893
14	3·7416573867739414
15	3·8729833460437668
17	4·1231056256176484
19	4·3657754179526917
21	4·5825756931198854
22	4·6904157598233230
23	4·7958315233127196
26	5·0990195135927848
29	5·3851648071345040

Factorial 'n'

$$1 \times 2 \times 3 \times 4 \times \ldots \times n = n!$$

n	$n!$
1	1
2	2
3	6
4	24
5	120
6	720
7	5040
8	40320
9	362880
10	3628800
11	39916800
12	479001600
13	6227020800
14	87178291200
15	1307674368000
16	20922789888000
17	355687428096000
18	6402373705728000
19	121645100408832000
20	2432902008176640000
21	51090942171709440000
22	1124000727777607680000
23	25852016738884976640000
24	620448401733239439360000
25	15511210043330985984000000
26	403291461126605635584000000
27	10888869450418352160768000000
28	304888344611713860501504000000
29	8841761993739701954543616000000
30	265252859812191058636308480000000
31	8222838654177922817725562880000000
32	263130836933693530167218012160000000
33	8683317618811886495518194401280000000
34	295232799039604140847618609643520000000

Triangular Numbers $n(n + 1)/2$

n	0	1	2	3	4	5	6	7	8	9
0	0	1	3	6	10	15	21	28	36	45
1	55	66	78	91	105	120	136	153	171	190
2	210	231	253	276	300	325	351	378	406	435
3	465	496	528	561	595	630	666	703	741	780
4	820	861	903	946	990	1035	1081	1128	1176	1225
5	1275	1326	1378	1431	1485	1540	1596	1653	1711	1770
6	1830	1891	1953	2016	2080	2145	2211	2278	2346	2415
7	2485	2556	2628	2701	2775	2850	2926	3003	3081	3160
8	3240	3321	3403	3486	3570	3655	3741	3828	3916	4005
9	4095	4186	4278	4371	4465	4560	4656	4753	4851	4950
10	5050	5151	5253	5356	5460	5565	5671	5778	5886	5995
11	6105	6216	6328	6441	6555	6670	6786	6903	7021	7140
12	7260	7381	7503	7626	7750	7875	8001	8128	8256	8385
13	8515	8646	8778	8911	9045	9180	9316	9453	9591	9730
14	9870	10011	10153	10296	10440	10585	10731	10878	11026	11175
15	11325	11476	11628	11781	11935	12090	12246	12403	12561	12720
16	12880	13041	13203	13366	13530	13695	13861	14028	14196	14365
17	14535	14706	14878	15051	15225	15400	15576	15753	15931	16110
18	16290	16471	16653	16836	17020	17205	17391	17578	17766	17955
19	18145	18336	18528	18721	18915	19110	19306	19503	19701	19900
20	20100	20301	20503	20706	20910	21115	21321	21528	21736	21945
21	22155	22366	22578	22791	23005	23220	23436	23653	23871	24090
22	24310	24531	24753	24976	25200	25425	25651	25878	26106	26335
23	26565	26796	27028	27261	27495	27730	27966	28203	28441	28680
24	28920	29161	29403	29646	29890	30135	30381	30628	30876	31125
25	31375	31626	31878	32131	32385	32640	32896	33153	33411	33670
26	33930	34191	34453	34716	34980	35245	35511	35778	36046	36315
27	36585	36856	37128	37401	37675	37950	38226	38503	38781	39060
28	39340	39621	39903	40186	40470	40755	41041	41328	41616	41905
29	42195	42486	42778	43071	43365	43660	43956	44253	44551	44850
30	45150	45451	45753	46056	46360	46665	46971	47278	47586	47895
31	48205	48516	48828	49141	49455	49770	50086	50403	50721	51040
32	51360	51681	52003	52326	52650	52975	53301	53628	53956	54285
33	54615	54946	55278	55611	55945	56280	56616	56953	57291	57630
34	57970	58311	58653	58996	59340	59685	60031	60378	60726	61075
35	61425	61776	62128	62481	62835	63190	63546	63903	64261	64620
36	64980	65341	65703	66066	66430	66795	67161	67528	67896	68265
37	68635	69006	69378	69751	70125	70500	70876	71253	71631	72010
38	72390	72771	73153	73536	73920	74305	74691	75078	75466	75855
39	76245	76636	77028	77421	77815	78210	78606	79003	79401	79800

n	0	1	2	3	4	5	6	7	8	9
40	80200	80601	81003	81406	81810	82215	82621	83028	83436	83845
41	84255	84666	85078	85491	85905	86320	86736	87153	87571	87990
42	88410	88831	89253	89676	90100	90525	90951	91378	91806	92235
43	92665	93096	93528	93961	94395	94830	95266	95703	96141	96580
44	97020	97461	97903	98346	98790	99235	99681	100128	100576	101025
45	101475	101926	102378	102831	103285	103740	104196	104653	105111	105570
46	106030	106491	106953	107416	107880	108345	108811	109278	109746	110215
47	110685	111156	111628	112101	112575	113050	113526	114003	114481	114960
48	115440	115921	116403	116886	117370	117855	118341	118828	119316	119805
49	120295	120786	121278	121771	122265	122760	123256	123753	124251	124750
50	125250	125751	126253	126756	127260	127765	128271	128778	129286	129795
51	130305	130816	131328	131841	132355	132870	133386	133903	134421	134940
52	135460	135981	136503	137026	137550	138075	138601	139128	139656	140185
53	140715	141246	141778	142311	142845	143380	143916	144453	144991	145530
54	146070	146611	147153	147696	148240	148785	149331	149878	150426	150975
55	151525	152076	152628	153181	153735	154290	154846	155403	155961	156520
56	157080	157641	158203	158766	159330	159895	160461	161028	161596	162165
57	162735	163306	163878	164451	165025	165600	166176	166753	167331	167910
58	168490	169071	169653	170236	170820	171405	171991	172578	173166	173755
59	174345	174936	175528	176121	176715	177310	177906	178503	179101	179700
60	180300	180901	181503	182106	182710	183315	183921	184528	185136	185745
61	186355	186966	187578	188191	188805	189420	190036	190653	191271	191890
62	192510	193131	193753	194376	195000	195625	196251	196878	197506	198135
63	198765	199396	200028	200661	201295	201930	202566	203203	203841	204480
64	205120	205761	206403	207046	207690	208335	208981	209628	210276	210925
65	211575	212226	212878	213531	214185	214840	215496	216153	216811	217470
66	218130	218791	219453	220116	220780	221445	222111	222778	223446	224115
67	224785	225456	226128	226801	227475	228150	228826	229503	230181	230860
68	231540	232221	232903	233586	234270	234955	235641	236328	237016	237705
69	238395	239086	239778	240471	241165	241860	242556	243253	243951	244650
70	245350	246051	246753	247456	248160	248865	249571	250278	250986	251695
71	252405	253116	253828	254541	255255	255970	256686	257403	258121	258840
72	259560	260281	261003	261726	262450	263175	263901	264628	265356	266085
73	266815	267546	268278	269011	269745	270480	271216	271953	272691	273430
74	274170	274911	275653	276396	277140	277885	278631	279378	280126	280875
75	281625	282376	283128	283881	284635	285390	286146	286903	287661	288420
76	289180	289941	290703	291466	292230	292995	293761	294528	295296	296065
77	296835	297606	298378	299151	299925	300700	301476	302253	303031	303810
78	304590	305371	306153	306936	307720	308505	309291	310078	310866	311655
79	312445	313236	314028	314821	315615	316410	317206	318003	318801	319600

n	0	1	2	3	4	5	6	7	8	9
80	320400	321201	322003	322806	323610	324415	325221	326028	326836	327645
81	328455	329266	330078	330891	331705	332520	333336	334153	334971	335790
82	336610	337431	338253	339076	339900	340725	341551	342378	343208	344035
83	344865	345696	346528	347361	348195	349030	349866	350703	351541	352380
84	353220	354061	354903	355746	356590	357435	358281	359128	359976	360825
85	361675	362526	363378	364231	365085	365940	366796	367653	368511	369370
86	370230	371091	371953	372816	373680	374545	375411	376278	377146	378015
87	378885	379756	380628	381501	382375	383250	384126	385003	385881	386760
88	387640	388521	389403	390286	391170	392055	392941	393828	394716	395605
89	396495	397386	398278	399171	400065	400960	401856	402753	403651	404550
90	405450	406351	407253	408156	409060	409965	410871	411778	412686	413595
91	414505	415416	416328	417241	418155	419070	419986	420903	421821	422740
92	423660	424581	425503	426426	427350	428275	429201	430128	431056	431985
93	432915	433846	434778	435711	436645	437580	438516	439453	440391	441330
94	442270	443211	444153	445096	446040	446985	447931	448878	449826	450775
95	451725	452676	453628	454581	455535	456490	457446	458403	459361	460320
96	461280	462241	463203	464166	465130	466095	467061	468028	468998	469965
97	470935	471906	472878	473851	474825	475800	476776	477753	478731	479710
98	480690	481671	482653	483636	484620	485605	486591	487578	488566	489555
99	490545	491536	492528	493521	494515	495510	496506	497503	498501	499500

A Few Specimen Primes

In making up numbers for testing factorisation methods it is essential to employ factors known to be primes. In a book of this kind it is neither necessary nor practicable to tabulate long lists of primes in sequence and the following selection should provide enough variety for most purposes.

113	401	797	2309	8803	13709	78901	107309	1000159	100004309
127	409	809	2311	8807	13711	78919	107339	1000171	100004327
131	419	811	2333	8819	13721	78929	107347	1000183	100004347
137	421	821	2339	8821	13723	78941	107351	1000187	100004363
139	431	823	2341	8831	13729	78977	107357	1000193	100004389
149	433	827	2347	8837	13751	78979	107377	1000199	100004393
151	439	829	2351	8839	13757	78989	107441	1000211	100004407
157	443	839	2357	8849	13759	79031	107449	1000213	100004417
163	449	853	2371	8861	13763	79037	107453	1000231	100004449
167	457	857	2377	8863	13781	79039	107467	1000249	100004461
173	461	859	2381	8867	13789	79043	107473	1000253	100004473
179	463	863	2383	8887	13799	79049	107507	1000273	100004477
181	467	877	2389	8893	13807	79063	107509	1000289	100004501
191	479	881	2393	8923	13829	79067	107533	1000291	100004503
193	487	883	2399	8929	13831	79103	107563	1000303	100004507
197	491	887	2411	8933	13841	79111	107581	1000313	100004519
199	499	907	2417	8941	13859	79133	107599	1000333	100004521
211	503	911	2423	8951	13873	79139	107603	1000357	100004533
223	509	919	2437	8963	13877	79147	107609	1000367	100004537
227	521	929	2441	8969	13879	79151	107617	1000381	100004549
229	523	937	2447	8971	13883	79153	107621	1000393	100004551
233	541	941	2459	8999	13901	79159	107641	1000397	100004561
239	547	947	2467	9001	13903	79181	107647	1000403	100004563
241	557	953	2473	9007	13907	79187	107671	1000409	100004629
251	563	967	2477	9011	13913	79193	107687	1000423	100004647
257	569	971	2503	9013	13921	79201	107693	1000427	100004651
263	571	977	2521	9029	13931	79229	107699	1000429	100004677
269	577	983	2531	9041	13933	79231	107713	1000453	100004719
271	587	991	2539	9043	13951	79241	107717	1000457	100004741
277	593	997	2543	9049	13963	79259	107719	1000507	100004813

n	2^n	3^n
2	4	9
3	8	27
4	16	81
5	32	243
6	64	729
7	128	2187
8	256	6561
9	512	19683
10	1024	59049
11	2048	177147
12	4096	531441
13	8192	1594323
14	16384	4782969
15	32768	14348907
16	65536	43046721
17	131072	129140163
18	262144	387420489
19	524288	1162261467
20	1048576	3486784401
21	2097152	10460353203
22	4194304	31381059609
23	8388608	94143178827
24	16777216	282429536481
25	33554432	847288609443
26	67108864	2541865828329
27	134217728	7625597484987
28	268435456	22876792454961
29	536870912	68630377364883
30	1073741824	205891132094649
31	2147483648	617673396283947
32	4294967296	1853020188851841
33	8589934592	5559060566555523
34	17179869184	16677181699666569
35	34359738368	50031545098999707
36	68719476736	150094635296999121
37	137438953472	450283905890997363
38	274877906944	1350851717672992089

n	2^n	3^n
39	549755813888	4052555153018976267
40	1099511627776	12157665459056928801
41	2199023255552	36472996377170786403
42	4398046511104	109418989131512359209
43	8796093022208	328256967394537077627
44	17592186044416	984770902183611232881
45	35184372088832	2954312706550833698643
46	70368744177664	8862938119652501095929
47	140737488355328	26588814358957503287787
48	281474976710656	79766443076872509863361
49	562949953421312	239299329230617529590083
50	1125899906842624	717897987691852588770249

ANSWERS TO THE EXERCISES

Chap. 1

1. 10101101100111.

2. 2(11)67.

3. (10)(27)7.

4. (*a*) The digits 2, (11), 6, 7, expressed in the scale of 2—allowing for the (legitimate) addition of some 0's—are respectively 10, 1011, 0110, 0111.

   (*b*) The digits (10), (27), 7, similarly expressed are 1010, 11011, 0111. Compare Ans. 1.

5.

123	456	456	~~123~~
61	912	228	~~246~~
30	~~1824~~	114	~~492~~
15	3648	57	984
7	7296	28	~~1968~~
3	14592	14	~~3936~~
1	29184	7	7872
	——	3	15744
	56088	1	31488
			——
			56088

6. Since $707 = 7 \times 101$, apply the '$s + 1$' rule after separating N into pairs of digits, thus: 45, 53, 31. Then $31 + 45 - 53 = 23$ = remainder.

Chap. 2

1. (*a*) × 13 and add.

```
21010101
      13
--------
 2101023
     39
--------
  210141
     13
--------
   21027
     91
--------
    2193
     39
--------
     258
    104
--------
     129 = 3 × 43.
```

```
10000000
 9837131
--------
  162869
       3
--------
  488607 = n
```

(*b*) × 30 and subtract.

```
21010101
      30
--------
 2100980
     240
--------
   20769
     270
--------
    1806
     180
--------
       0
```

```
  69801
      7
-------
= 488607
```

2.

```
110999999   (997 = 1000 − 3)
        3
---------
 11299999
        3
---------
  1329999
        3
---------
   332999
        9
---------
    33899
        9
---------
     3989   Therefore the remainder is 1,
        9   and the quotient is
      998   111333 + 1 = 111334.
      997
      ---
        1
```

3. Since $167 \times 6 = 1002$, use a similar procedure to the above but in this case *add* the multiples of 2.

Chap. 3

1. $3/41 = \cdot07317$ $\quad 29/41 = \cdot70731$
   $7/41 = \cdot17073$ $\quad 30/41 = \cdot73170$
   $13/41 = \cdot31707$

2. As with all recurrent cycles an algorithm—usually several—can always be found. I give the following because it differs markedly from those described in the text and shows something of the variety to be met with in this field.

   If n takes the successive values $1, 2, 3, \ldots n$, then $11(2^n - 2)$ defines the sequence 0, 22, 66, 154, 330, 682, 1386, etc. Dividing each term by 10^{2n} and summing, we have:

```
·00
  22
    66
     154
       330
        682
         1386
           2794
            5610
             11242
               22506
                 . . .   etc.
--------------------
·0022675736961451 24 . . .
```

   $11(2^n - 2)$ is, of course equivalent to $22(2^{n-1} - 1)$; the latter part of this will become quite familiar.

Chap. 4

1. 3·1416.

2. By definition

$$
\begin{aligned}
F_1 &= F_3 - F_2\\
F_2 &= F_4 - F_3\\
F_3 &= F_5 - F_4\\
&\;\;\ldots\\
F_{n-1} &= F_{n+1} - F_n\\
F_n &= F_{n+2} - F_{n+1}
\end{aligned}
$$

Adding all these equations together we have on the left hand side the sum of the first Fibonacci numbers, whilst the terms on the right sum to $F_{n+2} - F_2$.

Since $F_2 = 1$, the required sum is therefore $F_{n+2} - 1$.

Chap. 6

1. $1001001001 \times 111 = I_{12}$
   $100010001 \times 1111 = I_{12}$

   The factors common to both numbers are therefore those of I_{12} omitting those of I_3 and I_4, (3, 11, 37, 101). Hence they are 7, 13, *and* 9901.

2. (See Tables 8 and 9)
   The sequence 91, 9091, 909091, 90909091, etc. occurs in the factorisations of I_6, I_{10}, I_{14}, . . . I_{4k+1}. Since 90909091 does not appear among the prime factors of I_{18} it must therefore be composite.

3. $I_{12} = 3333 \times 33336667 = 3.7.11.13.37.101.9901$
   Since $3333 = 3.11.101$,
   then $33336667 = 7.13.37.9901$.

Chap. 7

1. The eight numbers relatively prime to 20 are 1, 3, 7, 9, 11, 13, 17, and 19 and therefore by Euler's generalisation n can take any of the values $20k + 1, 3, 7, 9$, etc.
   Thus, for instance $7^8 - 1 = 5764800$
   and $19^8 - 1 = 16983563040$.

2. The index 12 appears at the following places in the table: $m = 13, 21, 26, 28, 36, 42$. Collectively these numbers contain the primes 2, 3, 7, and 13 and it follows that these are all factors of $5^{12} - 1$.
   (5 being relatively prime to 12).

3. $402 = 2.3.67$.
   $$\begin{aligned}\phi(402) &= 402(1 - 1/2)(1 - 1/3)(1 - 1/67)\\ &= 402 \times 1/2 \times 2/3 \times 66/67\\ &= 6 \times 1/3 \times 66 = 2 \times 66 = 132.\end{aligned}$$

Chap. 9

1. $$5x = 1001 - 9y$$
   Remainders on dividing by 5, 1 4
   Multiplying the integers 1 2 3 4
   by 4, removing multiples of 5; 4 3 2 1
   1 is in the fourth position, hence $y = 4$.
   Then $5x = 1001 - 36 = 965$. $x = 193$.

2. For $z = 26 = m^2 + n^2$, the only positive integer values of m and n are 5 and 1.
   Then $x = m^2 - n^2 = 24$
   and $y = 2mn = 10$
   i.e. $24^2 + 10^2 = 26^2$.

Chap. 10

1. (*a*) 1. (*b*) 5. (Extrapolated from Tables 14*a*, *b*).

2. Divide each term as it appears, by 7; the remainder multiplied by three then provides the next term. In this way we get the sequence 1, 3, 2, 6, 4, 5, 1, 3, etc. The remainder thus appears as 1 when $n = 0, 6, 12, \ldots = 6k$ and therefore $3^{6k} - 1$ is always divisible by 7.

3. 5 and 8. (See Table 14*b*.)

Chap. 11

1. $99999999 = 10^8 - 1$
   $$\begin{array}{ll}\text{Now} & 10^2 \equiv -1 \pmod{101}\\ \text{Therefore} & 10^8 \equiv 1\\ \text{or} & 10^8 - 1 \equiv 0 \pmod{101}\end{array}$$

2. $$\begin{aligned}2^5 &= 32 \equiv 9 \pmod{23}\\ 2^{10} &\equiv 81 \equiv 12\\ 2^{11} &\equiv 24 \equiv 1\\ 2^{22} &\equiv 1. \text{ or } 2^{22} - 1 \equiv 0 \pmod{23}.\end{aligned}$$

3. Dividing throughout by 7 we find that 2908456 has the remainder 5, which is not a quadratic residue of 7. (*b*) is therefore not a square. Dividing *a*, *c*, *d*, *e*, by 11, (*e*) has the remainder 2, and hence is not a square.
   Ans. (*b*) and (*e*).

Chap. 12

1. 29.

```
16733)35699
       33466
       2233)16733
            15631
             1102)2233
                  2204
                    29)1102
                        87
                        232
                        232
                          0
```

2. It is easily seen that 3, 7, 11, 13, are not divisors so we set up

$$\underset{17\leq p\leq 29}{\Pi(p)} = 17.19.23.29 = 215441.$$

```
215441)1181027
       1077205
       103822)215441
              207644
                7797)103822
                      7797
                      25852
                      23391
                       2461)7797
                            7383
                             414)2461
                                 2484
                               -   23
```

Therefore 23 is a factor.

3.

```
         1 0 4 6²
     N = 1093709
         100
 204      937
          816
2086     12109
         12516
           407
          2093 = (2 × 1046) + 1
          2500 = 50²
```

Therefore $N = 1047^2 - 50^2 = 1097 \times 997.$

Chap. 13

1.
$15871 = 59 \times 269.$
$15853 = 83 \times 191.$
$15863 = 29 \times 547.$

2.
$13333 = 67 \times 199.$
$548497 = 53 \times 79 \times 131.$

by D. St. P. Barnard

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NUMBERS

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